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# MATHEMATICS magazine

# MATHEMATICS MAGAZINE

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*Address* editorial correspondence to Glenn James, special papers to the editors of the departments for which they are intended, and general papers to a member of the executive committee.

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**MATHEMATICS MAGAZINE**

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The annual report to Editors and Sponsors will be sent out in July after the close of the fiscal year. Editor.

# OUR CONTRIBUTORS

*H. Kaufman*, Associate Professor of Mathematics at McGill University, Montreal, Canada, was born in Lachine, Quebec in 1920 and received his formal education at McGill University (B.Sc.'41, M.Sc. '46, Ph.D.'48). A specialist in applied mathematics and physics, Dr. Kaufman worked as a research geophysicist, Continental Oil Co. and as a physicist for the Laboratory for Electrons, Inc. before being appointed to his present position in 1952.

*S. B. Jackson* is Professor of Mathematics and Department Chairman at the University of Maryland. Born in Wiscasset, Maine in 1913, he attended Bates College (B.A.'33) and did his graduate work at Harvard University (A.M.'34, Ph.D.'37). After teaching at the University of Wisconsin from 1937-41 he joined the faculty at Maryland as an Assistant Professor, and was promoted to a Professorship in 1948. His particular mathematical interests lie in geometry, especially differential geometry in the large.

*D. Greenspan* was a student at the University of Maryland when he collaborated with Jackson on the paper on hyberbolic geometry.

*H. W. Becker* was born in Sioux County, Nebraska in 1908. After attending the University of Chicago and the University of California, he went into business in Omaha. Lately Mr. Becker has been working as a "classical disc jockey" and engineer at the KSWI and WOW radio and television stations at Council Bluffs and Omaha. He is also engaged in preparing a concordance to the second volume of Dickson's *HISTORY OF THE THEORY OF NUMBERS*.

*B. S. Bissinger* is Chairman of the Department of Mathematics, Lebanon Valley College. A graduate of Franklin and Marshall College (B.S.'38), Syracuse University (M.A.'40) and Cornell University (Ph.D.'43), he served during the war on the Applied Mathematics Panel, Nat. Res. Council and was also a member of the Operations Analysis Section, A.A.F., attached to General Chennault for a year in China. After the war he became executive vice-president of a shoe corporation. He still retains contact with the business world as a consultant. (*THE EGG PROBLEM*, Jan.-Feb.)

*Jerome Hines* is a distinguished basso at the Metropolitan Opera, New York and is often heard on radio and television. A graduate of the University of California, Los Angeles, with a major in mathematics, he is still keenly interested in the subject, as his article *OPERATOR MATHEMATICS* in the last issue indicates.

(Continued on inside of the back cover.)

# HARMONIC DISTORTION IN POWER-LAW DEVICES

H. Kaufman

*Foreword*

An analysis is made of a biased power-law device with a single frequency input. The exponent of the power describing the device is an arbitrary, non-negative, real number. Recurrence relations are derived for the harmonics in the output of a given device, relations are given between harmonics produced by devices with different power-law characteristics, and harmonics associated with positive values of bias are related to those associated with negative values of bias. The methods used depend on elementary properties only of integration. For convenience of reference some scattered results from the literature are brought together.

*Introduction*

The harmonic analysis of the output of a nonlinear device obeying a power-law characteristic, with a single frequency input, has been the subject of numerous investigations. In particular, explicit formulas for the harmonics in terms of hypergeometric functions have been given by Bennett[1] and Lampard[2]. Salzberg[3] treats certain special cases and gives some references to earlier work. During the course of an investigation into the modulation product problem for a two-frequency input, some simple recurrence relations for the single-frequency case were derived which seem to have been overlooked by other authors. Harnwell[4] gives a recurrence relation for the zero-bias case, and essentially the same relation is rederived by Rawcliffe in an appendix to a paper by Slemon[5].

The chief objects of the present paper are to derive recurrence relations for the harmonics in the output of a given biased power-law device, to relate the harmonics produced by devices with different power-law characteristics, and to relate the harmonics associated with positive values of bias to those associated with negative values of bias. In addition some scattered results from the literature are brought together. The notation for the most part is based on that of Sternberg and Kaufman[6].

## ANALYSIS OF POWER-LAW CHARACTERISTIC

Consider a device whose output versus input characteristic is given by

(1) $$Y(X;X_0) = \begin{cases} (X-X_0)^v, & X > X_0 \\ 0, & X \le X_0 \end{cases}$$

where $v$ is a nonnegative real number. Let the input be $x(t) = P\cos(pt + \theta_p)$. Then with the output $y(t) \equiv Y(x(t);X_0)$ is associated a Fourier series expansion

(2) $$y(t) = \tfrac{1}{2}C_0 + \sum_{m=1}^{\infty} C_m \cos(mpt + m\theta_p)$$

where

(3) $$C_m = (1/\pi)\int_{-\pi}^{\pi} Y(P\cos u;\ X_0)\cos mu\ du$$

This can be written

(4) $$C_m = P^v A_m^{(v)}$$

where

(5) $$A_m^{(v)} = (1/\pi)\int(\cos u - h)^v \cos mu\ du\quad (m = 0,1,2,\cdots)$$

and $h \equiv X_0/P$. The limits of integration in (5) depend on the values of $h$, and are specified below in (7) and (7a). The quantity $h$ can be considered as a bias normalized with respect to input amplitude.

Three distinct cases must be considered, depending on the different values of $h$:

(6)
$$\begin{aligned} &\text{case } (o): && h \ge 1 \\ &\text{case } (i): && -1 < h < 1 \\ &\text{case } (\infty): && h \le -1 \end{aligned}$$

In case $(o)$, operation is entirely on the nonconducting portion of the characteristic (to borrow from electrical terminology) so that

$$A_m^{(v)} = 0,\quad (m = 0,1,2,\cdots).$$

In case $(i)$, operation is partly on the conducting and partly on the nonconducting portions of the characteristic:

(7) $$A_m^{(v)} = (2/\pi)\int_0^{\alpha}(\cos u - h)^v \cos mu\ du,\quad (m = 0,1,2,\cdots)$$

where $\alpha = \cos^{-1}h$.

In case $(\infty)$, operation is entirely on the conducting portion of the characteristic:

(7a) $A_m^{(v)} = (2/\pi)\int_0^{\pi}(\cos u - h)^v \cos mu\, du, \quad (m = 0,1,2,\cdots)$

Salzberg[3] refers to cases $(i)$ and $(\infty)$ as those of discontinuous current and continuous current, respectively.

## EXPLICIT RESULTS OF BENNETT AND LAMPARD

For case $(i)$, Bennett gives an explicit formula for $A_m^{(v)}$ in terms of the Gaussian hypergeometric function $F(a,b;c;x)$:

$$(8)\quad A_m^{(v)} = (2/\pi)^{\frac{1}{2}} \left[\frac{\Gamma(v+1)}{\Gamma(v+3/2)}\right]\cdot(1-h)^{v+\frac{1}{2}}\cdot F(\tfrac{1}{2}+m,\ \tfrac{1}{2}-m;\ v+3/2;\ \tfrac{1}{2}(1-h))$$

where $\Gamma(\ )$ denotes the Gamma function.

In the zero-bias case $(h=0)$ this reduces to

$$(8a)\quad A_m^{(v)} = (2/\pi)^{\frac{1}{2}}\ \frac{\Gamma(v+1)\Gamma(\frac{2v+3}{4})\Gamma(\frac{2v+5}{4})}{\Gamma(v+\frac{3}{2})\Gamma(\frac{2+v+m}{2})\Gamma(\frac{2+v-m}{2})}$$

For case $(\infty)$, Lampard derives the following expression

$$(9)\quad A_m^{(v)} = \left[\frac{(-h)^{v-m}}{2^{m-1}m!}\right]\left[\frac{\Gamma(v+1)}{\Gamma(v-m+1)}\right]\cdot F[\tfrac{1}{2}(m-v),\ \tfrac{1}{2}(m-v+1);\ m+1;\ h^{-2}]$$

or, in terms of the associated Legendre function of the first kind

$$(10)\quad A_m^{(v)} = 2(-h)^v(1-h)^{v/2}\left[\frac{\Gamma(v+1)}{\Gamma(v+m+1)}\right] P_v^m[(1-h^{-2})^{-\frac{1}{2}}]$$

For cases $(i)$ and $(\infty)$ Salzberg[3] considers the special types $v=1, 3/2, 2$.

## RECURRENCE RELATIONS

We now derive an expression relating three successive harmonics $A_{m-1}^{(v)}$, $A_m^{(v)}$, $A_{m+1}^{(v)}$, the value of $v$ being kept fixed. The proof is entirely elementary, and avoids the rather cumbersome relations between contiguous hypergeometric functions which would be required if the explicit forms (8) or (9) were used.

For case $(i)$ we have

$$A_{m\pm1}^{(v)} = (2/\pi)\int_0^{\alpha}(\cos u - h)^v(\cos mu\cos u \mp \sin mu \sin u)\,du \quad (m \pm 1 \geq 0)$$

whence

$$(m-v-1)A_{m-1}^{(v)} + (m+v+1)A_{m+1}^{(v)} = (4m/\pi)\int_0^{\alpha}(\cos u - h)^v \cos mu \cos u \, du -$$
$$(v+1)(4/\pi)\int_0^{\alpha}(\cos u - h)^v \sin mu \sin u \, du$$

The first term on the right reduces to

$$2mhA_m^{(v)} + (4m/\pi)\int_0^{\alpha}(\cos u - h)^{v+1}\cos mu \, du$$

The second term on the right reduces to

$$-(4m/\pi)\int_0^{\alpha}(\cos u - h)^{v+1}\cos mu \, du$$

on integrating by parts; whence

(11) $$(m+v+1)A_{m+1}^{(v)} - 2mhA_m^{(v)} + (m-v-1)A_{m-1}^{(v)} = 0 \qquad (m = 1,2,3,\cdots)$$

Thus $A_m^{(v)}$ $(m \geq 2)$ can be evaluated in terms of $A_0^{(v)}$ and $A_1^{(v)}$.

The proof for case (∞) is similar, with limits 0 to $\alpha$ replaced by 0 to $\pi$.

The relation given by Harnwell[4] and Rawcliffe[5] is the special form of (11) with $h = 0$.

## RELATIONS BETWEEN HARMONICS FOR DIFFERENT POWER-LAW DEVICES

In this section we relate the harmonics of a $(v+1)^{th}$- law device to those for a $v^{th}$- law device. By definition for case ($i$)

$$\begin{aligned} A_m^{(v+1)} &= (2/\pi)\int_0^{\alpha}(\cos u - h)^{v+1}\cos mu \, du, \quad (m = 0,1,2,\cdots) \\ &= (2/\pi)\int_0^{\alpha}(\cos u - h)^{v}(\cos u - h)\cos mu \, du \\ &= (2/\pi)\int_0^{\alpha}(\cos u - h)^{v}\tfrac{1}{2}[\cos(m+1)u + \cos(m-1)u]\, du \\ &\quad -(2h/\pi)\int_0^{\alpha}(\cos u - h)^{v}\cos mu \, du, \quad (m = 1,2,3,\cdots) \end{aligned}$$

whence

(12) $$2A_m^{(v+1)} = A_{m+1}^{(v)} - 2hA_m^{(v)} + A_{m-1}^{(v)}, \quad (m = 1,2,3,\cdots)$$

Eliminating $A_m^{(v)}$ from (11) and (12) we obtain

(13) $$2mA_m^{(v+1)} = (v+1)(A_{m-1}^{(v)} - A_{m+1}^{(v)}), \quad (m = 1,2,3,\cdots)$$

Similarly

(13a) $$A_0^{(v+1)} = A_1^{(v)} - hA_0^{(v)}$$

The proof for case (∞) can be carried out in the same way.

By means of formulas (13) and (13a) the values of $A_m^{(v)}$ for integral $v$ can be derived very simply from those for $v = 0$. More generally, if the computation of $A_m^{(v)}$ is carried out for a sufficiently large selection of values of $v$ between zero and one, then (13) and (13a) provide a simple means of computing $A_m^{(v)}$ for $v \geq 1$ without recourse to the defining integrals.

For $v = 0$ we have

$$A_0^{(0)} = 2\alpha/\pi \quad A_m^{(0)} = (2/m\pi)\sin m\alpha, \quad (m \geq 1), \text{ in case } (i)$$

$$A_0^{(0)} = 2, \qquad A_m^{(0)} = 0, \qquad (m \geq 1), \text{ in case } (\infty).$$

The $0^{th}$-law device is also known as a total limiter (Bennett[1]).

Formulas (13) and (13a) provide a simple proof that for integral $v$ in case $(\infty)$, $A_m^{(v)}=0$, $(m = v+1,\ v+2,\ v+3, \cdots)$.

## REFLECTION RELATIONS

By a reflection relation we understand an expression relating harmonics associated with positive values of $h$ to those associated with negative values of $h$. Let $h$ be positive and in case $(i)$ (the degenerate case $(o)$ is of no interest here), then $-h$ is also in case $(i)$.

$$A_m^{(v)}(-h) = (2/\pi)\int_0^{\cos^{-1}(-h)} (\cos u + h)^v \cos mu \, du$$
$$= (2/\pi)\left(\int_0^{\pi} - \int_{\cos^{-1}(-h)}^{\pi}\right)(\cos u + h)^v \cos mu \, du.$$

Making the substitution $u = \pi - u'$ in the second integral on the right we have

$$\int_{\cos^{-1}(-h)}^{\pi} (\cos u + h)^v \cos mu \, du = (-1)^{m+v}\int_0^{\cos^{-1} h} (\cos u' - h) \cos mu' \, du'$$

whence

$$(14) \quad A_m^{(v)}(-h) = \hat{A}_m^{(v)}(-h) + (-1)^{m+v+1} A_m^{(v)}(h), \quad (m = 0,1,2,\cdots)$$

where

$$\hat{A}_m^{(v)}(-h) = (2/\pi)\int_0^{\pi} (\cos u + h)^v \cos mu \, du.$$

Note that $\hat{A}_m^{(v)}(-h)$ is not in case $(\infty)$, despite the limits 0 to $\pi$, since $-1 < h < 1$.

The above reflection relation admits of ready generalization to more complicated types of characteristics. Thus if

$$A_m(h) = (2/\pi)\int_0^{\alpha} f(\cos u - h)\cos mu \, du \qquad (\text{case } (i))$$

where $f(z)$ is either an even, or an odd function of $z$; i.e., $f(-z) = \delta f(z)$ where $\delta = +1$ or $-1$ according as $f(z)$ is even or odd, then

$$A_m(-h) = \hat{A}_m(-h) + (-1)^{m+1}\delta A_m(h), \quad (m = 0,1,2,\cdots) \tag{14a}$$

where

$$\hat{A}_m(-h) = (2/\pi)\int_0^{\pi} f(\cos u + h)\cos mu\, du.$$

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5. G. R. Slemon, *A Method of Approximate Steady-State Analysis for Non-Linear Networks*, with an Appendix by G. H. Rawcliffe, Proc. I.E.E., 100, Part I, 275-287 (1953).
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McGill University, Montreal, Que.

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## Fellowship For Secondary School Mathematics Teachers

Cleveland, Ohio, March 1, 1955: Announcement of thirty all-expenses paid fellowships for secondary school mathematics teachers to attend a program being held at Case Institute of Technology from June 19 to July 29, 1955 was made today by Dean Elmer Hutchisson.

The program, being sponsored in 1955 by the E. I. du Pont de Nemours Company, Inc., is designed especially for the Fellows and will be taught by members of the Case Faculty.

Thirty fellowships covering tuition, board and lodging, books and supplies, and travel expenses to and from Cleveland are to be awarded to qualified secondary school teachers. In this way it is hoped to provide recognition for outstanding contributions to mathematical education and to stimulate future efforts in this area of knowledge.

Al Henderson
Director of Public Relations

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## In Our Next Issue

An understandable treatment of *Combinatorial Topology of Surfaces* by Robert C. James, based on lectures given by Prof. A. W. Tucker of Princeton University while a Philips visitor at Haverford College, will appear in our Sept.-Oct. issue. Editor.

# HYPERBOLIC ANALYTIC GEOMETRY

Stanley B. Jackson and Donald Greenspan

## 1. *Introduction*

The individual who encounters hyperbolic geometry for the first time in such a book as the one by Wolfe [3] has the stimulating experience of developing the analogue of a substantial part of euclidean geometry using the same essential spirit and methods as those of Euclid. This is followed by a development of hyperbolic trigonometry which provides the reader with all the rudimentary data for solving triangle problems in the hyperbolic plane. However, only a few texts, like the one of Sommerville [2], make an attempt to develop a hyperbolic analytic geometry in a way which parallels the usual freshman course in analytics, and the student has only to attempt as simple a problem as the determination of the equation of the altitude of an arbitrary triangle to realize the inadequacy of his equipment. Other texts, such as that by Coxeter [1] approach hyperbolic geometry from the viewpoint of projective geometry. To a student familiar with projective geometry it is then a relatively simple matter to think of the hyperbolic plane as embedded in the projective plane and develop its analytic geometry in the usual projective coordinates. From the beginner's point of view it seems an unnatural way of getting analytic results for the hyperbolic plane.

The aim of this paper is to present a brief development of hyperbolic analytic geometry following the usual procedures of analytics quite familiar to students. Only a single use is made of calculus, and this could probably have been avoided. From the discussion follows naturally the idea of introducing a different coordinate system which amounts in fact to embedding in the projective plane. Thus the procedure of establishing hyperbolic geometry as a subgeometry of projective geometry is obtained naturally from within in an elementary way. With the exception of the last section, which utilizes a little projective geometry and the already noted single use of calculus, nothing is assumed beyond analytic geometry and an introduction to hyperbolic geometry.

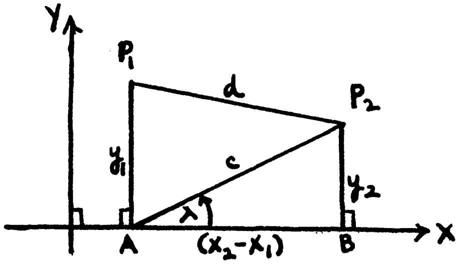
The question of applicability of the results and methods to binocular sensory space [4] remains to be examined.

## 2. *Distance between two points*

Let two perpendicular directed lines be taken for the coordinate axes. The coordinates $(x,y)$ of an arbitrary point $P$ are defined as follows: $x$ denotes the directed distance from the origin to the foot

$D$ of the perpendicular from $P$ on the $x$-axis, while $y$ is the directed distance $DP$. For simplicity the unit of distance in the geometry is chosen so that the constant $k$ for the geometry is unity [3, p. 153]. For general choice of unit it is only necessary to replace each distance $a$ by $a/k$.

Consider two distinct points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ and suppose first that $x_2 \geq x_1$ and $y_1$ and $y_2$ both non-negative. Let $d$ be the required distance and draw $c$ from $A$ to $P_2$ as shown.




From right triangle $ABP_2$ it follows [3, p. 153] that

$$\cosh c = \cosh y_2 \cosh (x_2 - x_1)$$

$$\sinh y_2 = \sinh c \sin \lambda,$$

while the hyperbolic law of cosines, applied to triangle $AP_1P_2$ yields

$$\cosh d = \cosh y_1 \cosh c - \sinh y_1 \sinh c \cos(\pi/2 - \lambda).$$

By substitution from above this becomes

$$(2.1) \qquad \cosh d = \cosh y_1 \cosh y_2 \cosh(x_2 - x_1) - \sinh y_1 \sinh y_2.$$

The argument is readily modified to show that (2.1) holds whatever the choice of signs for $y_1$ and $y_2$, and since the formula is symmetric in the points $P_1$ and $P_2$ the condition $x_2 \geq x_1$ is clearly unnecessary.

The equation of the perpendicular bisector of a segment $P_1P_2$ may be found in the usual way by equating distances $P_1P$ ans $P_2P$ where $P$ is the moving point $(x, y)$. This equation, after division by $\cosh y$, becomes

$$(2.2) \qquad (\sinh y_2 - \sinh y_1) \tanh y + (\sinh x_2 \cosh y_2 - \sinh x_1 \cosh y_1) \sinh x + (\cosh y_1 \cosh x_1 - \cosh y_2 \cosh x_2) \cosh x = 0.$$

## 3. *The straight line*

Equation (2.2) just derived has the form

(3.1) $a_1 \tanh y + a_2 \sinh x + a_3 \cosh x = 0, \quad a_1^2 + a_2^2 + a_3^2 \neq 0$

Moreover, since every line is the perpendicular bisector of some suitably chosen segment, it follows that every line has an equation of the form (3.1). Conversely, consider an equation of form (3.1) and suppose first that $a_3 \neq 0$ so that the origin does not satisfy (3.1). Equation (3.1) will be a line if and only if for any point $(x_1, y_1)$ not satisfying the equation, there can be found a point $(x_2, y_2)$ so that the coefficients of (2.2) are proportional to those of (3.1). Taking $(x_1, y_1)$ as the origin, the required equations become

$$\sinh y_2 = m\, a_1$$
$$\sinh x_2 \cosh y_2 = m\, a_2$$
$$\cosh x_2 \cosh y_2 = 1 - m\, a_3$$

where $m$ is the proportionality factor. From the first two equations and the identity $\cosh^2 z - \sinh^2 z = 1$, it is seen that

$$\cosh y_2 = \sqrt{1 + m^2 a_1^2}$$

$$\cosh x_2 = \sqrt{1 + \sinh^2 x_2} = \sqrt{\frac{1 + m^2 a_1^2 + m^2 a_2^2}{1 + m^2 a_1^2}}$$

Substitution in the third equation yields

(3.2) $$\sqrt{1 + m^2 a_1^2 + m^2 a_2^2} = 1 - m a_3$$

Squaring and solving for $m$ we find, since $m = 0$ has no meaning, that the formal solution is

$$m = \frac{-2a_3}{a_1^2 + a_2^2 - a_3^2}$$

However, this formal solution satisfies (3.2) if and only if $1 - m a_3 > 0$, which is equivalent to

(3.3) $$a_1^2 + a_2^2 - a_3^2 > 0.$$

Thus, if $a_3 \neq 0$, (3.1) is a line if and only if (3.3) holds.

If, on the other hand, $a_3 = 0$, then (3.1) becomes $\frac{\tanh y}{\sinh x} = -\frac{a_2}{a_1}$. But if the line $OP$ is drawn from the origin to the point $P(x, y)$ and if $OP$ denotes the angle from the $x$-axis to the line $OP$, it follows easily from the formulas for right triangles [3, p.153] that

$$\frac{\tanh y}{\sinh x} = \tan \lambda .$$

Hence for all points $(x,y)$ satisfying (3.1) with $a_3 = 0$ it follows that $\tan \lambda = -a_2/a_1$. Thus $\lambda$ is constant, and (3.1) is an equation of the line through the origin making angle Arctan $(-a_2/a_1)$ with the $x$-axis. Since for this case (3.3) is automatically satisfied, (3.3) is a necessary and sufficient condition that (3.1) represent a line.

If (3.3) is false then (3.1) has no locus. Assume that (3.3) is false so that $a_3^2 \geq a_1^2 + a_2^2$. If $a_3^2 > a_2^2$ let $b$ be defined by the equations

$$\cosh b \equiv \frac{\epsilon a_3}{\sqrt{a_3^2 - a_2^2}}, \quad \sinh b = \frac{-\epsilon a_2}{\sqrt{a_3^2 - a_2^2}}$$

where $\epsilon = \pm 1$ and is chose so that $\epsilon a_3 > 0$. Then, for any point $(x,y)$

$$|a_1 \tanh y + a_2 \sinh x + a_3 \cosh x| \geq |a_2 \sinh x + a_3 \cosh x| - |a_1 \tanh y|$$
$$= \sqrt{a_3^2 - a_2^2} \cosh (x - b) - |a_1 \tanh y| \geq \sqrt{a_3^2 - a_2^2} - |a_1 \tanh y| >$$
$$|a_1| - |a_1 \tanh y| \geq 0,$$

where use has been made of the fact that $|\tanh y| < 1$ and $(x-b) \geq 1$, as well as of (3.3). Thus no point $(x,y)$ can satisfy (3.1). If, on the other hand, $a_3^2 = a_2^2$, then $a_2 = \pm a_3$ and $a_1 = 0$, so that (3.1) becomes either $a_3 e^x = 0$ or $a_3 e^{-x} = 0$, neither of which has any solution. These results may be summarized as follows.

**Theorem 3. 1.** *Every line has an equation of form* (3.1). *Conversely, every equation of form* (3.1) *is the equation of a line if* $a_1^2 + a_2^2 - a_3^2 > 0$ *and has no locus if* $a_1^2 + a_2^2 - a_3^2 \leq 0$.

In order for a line (3.1) to be parallel to the x-axis in the positive direction it is necessary and sufficient that $\lim_{x \to \infty} (a_2 \sinh x + a_3 \cosh x) = 0$. But since

$$a_2 \sinh x + a_3 \cosh x = \frac{(a_3 + a_2)}{2} e^x + \frac{(a_3 - a_2)}{2} e^{-x}$$

it follows that this holds if and only if $a_3 = -a_2$. Similarly parallelism to the $x$-axis in the negative direction occurs if and only if $a_3 = a_2$. Thus,

**Theorem 3.2** *A line (3.1) is parallel to the $x$-axis in the positive (negative) sense if and only if $a_3 + a_2 = 0 (a_3 - a_2 = 0)$.*

## 4. *Intersections of lines*

Let the lines represented by the equations

$$a_1 \tanh y + a_2 \sinh x + a_3 \cosh x = 0, \quad b_1 \tanh y + b_2 \sinh x + b_3 \cosh x = 0,$$

be referred to respectively as the lines **a** and **b**, and for brevity let the determinant $\begin{vmatrix} a_i & a_j \\ b_i & b_j \end{vmatrix}$ be represented by the symbol $|a_i b_j|$.

If **a** and **b** are lines whose equations are not proportional, an attempt to solve simultaneously leads at once to the equations

$$\tanh y = p|a_2 b_3|, \quad \sinh x = p|a_3 b_1|, \quad \cosh x = p|a_1 b_2|.$$

Since the proportionality factor $p$ is uniquely determined by the relations $\cosh^2 x - \sinh^2 x = 1$ and $\cosh x > 0$, these equations become

$$\tanh y = \frac{\epsilon |a_2 b_3|}{\sqrt{|a_1 b_2|^2 - |a_3 b_1|^2}}, \tag{4.1}$$

$$\sinh x = \frac{\epsilon |a_3 b_1|}{\sqrt{|a_1 b_2|^2 - |a_3 b_1|^2}}, \quad \cosh x = \frac{\epsilon |a_1 b_2|}{\sqrt{|a_1 b_2|^2 - |a_3 b_1|^2}},$$

where $\epsilon = \pm 1$ and is chosen so that $\epsilon |a_1 b_2| > 0$. In order that there be a point $(x, y)$ defined by equations (4.1) it is necessay and sufficient that the denominators be real and different from zero and that the expression for $\tanh y$ be numerically less than 1. These two conditions are equivalent to the inequality

$$|a_1 b_2|^2 - |a_2 b_3|^2 - |a_3 b_1|^2 > 0. \tag{4.2}$$

This establishes the following result.

**Theorem 4.1.** *A necessary and sufficient condition that distinct lines* **a** *and* **b** *intersect is that they satisfy* (4.2) *The point of intersection is then given by* (4.1).

Incidentally, since the point $(x,y)$ given by (4.1) is unique if it exists at all, two non-proportional equations of lines **a** and **b** cannot represent the same line, so equation (3.1) for a given line is unique to within a proportionality factor.

From (4.1) it is easy to derive the equivalent formulas

$$(4.3)\quad \sinh y = \frac{\epsilon|a_2b_3|}{\Delta},\quad \cosh y = \frac{\sqrt{|a_1b_2|^2 - |a_3b_1|^2}}{\Delta},\quad \tanh x = \frac{|a_3b_1|}{|a_1b_2|}$$

where $\Delta$ is defined by the equation

$$\Delta = \sqrt{|a_1b_2|^2 - |a_2b_3|^2 - |a_3b_1|^2}\ .$$

It may be noted at once that the equation of the line joining distinct points $(x_1,y_1)$ and $(x_2,y_2)$ may be written in the form

$$(4.4)\qquad \begin{vmatrix} \tanh y & \sinh x & \cosh x \\ \tanh y_1 & \sinh x_1 & \cosh x_1 \\ \tanh y_2 & \sinh x_2 & \cosh x_2 \end{vmatrix} = 0.$$

## 5. *Algebraic lemma*

Lemma 5.1. $|a_1b_2|^2 - |a_2b_3|^2 - |a_3b_1|^2 \equiv$

$$(a_1^2 + a_2^2 - a_3^2)(b_1^2 + b_2^2 - b_3^2) - (a_1b_1 + a_2b_2 - a_3b_3)^2\ .$$

This is readily verified by direct expansion, or by substitution of $a_3i$ and $b_3i$ for $a_3$ and $b_3$ in the familiar LaGrange's identity, i.e.,

$$|a_1b_2|^2 + |a_2b_3|^2 + |a_3b_1|^2 = (a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2) - (a_1b_1+a_2b_2+a_3b_3)^2.$$

## 6. *Parallelism*

**Theorem 6.1.** *A necessary and sufficient condition that two distinct lines,* **a** *and* **b**, *be parallel is that*

$$(6.1)\qquad |a_1b_2|^2 - |a_2b_3|^2 - |a_3b_1|^2 = 0.$$

By Theorem 3.2, a necessary and sufficient condition that lines **a** and **b** be parallel in the same sense to the $x$-axis is that $a_3 = \epsilon a_2$ snd $b_3 = \epsilon b_3$ where $\epsilon^2 = 1$. This condition implies that $|a_2 b_3| = 0$ and, using Lemma 5.1, that (6.1) holds. Suppose, conversely, that (6.1) holds and that $|a_2 b_3| = 0$. Note that it is not possible to have $a_2 = b_2 = 0$ for then, by (6.1), **a** and **b** would have proportional equations. Thus there exists an $\epsilon$ so that $a_3 = \epsilon a_2$ and $b_3 = \epsilon b_2$, and substitution in (6.1) yields at once $|a_1 b_2|^2(1 - \epsilon^2) = 0$. But since **a** and **b** are distinct, their equations are not proportional and $|a_1 b_2| \neq 0$, whence $\epsilon^2 = 1$. Thus the conditions are sufficient. This disposes of the case of parallelism with the $x$-axis.

For two lines **a** and **b**, not parallel to the $x$-axis, to be parallel to each other it is necessary and sufficient that they have a common parallel which is perpendicular to the $x$-axis, that is that there shall be a value $x = \bar{x}$ so that as $x$ approaches $\bar{x}$ the values of $y$ for the two lines shall become infinite in the same sense. This is equivalent to asking that for both lines $\lim_{x \to \bar{x}} \tanh y = \epsilon_1$ where $\epsilon_1^2 = 1$, which in turn is equivalent to demanding that $\bar{x}$ satisfy the equations

$$(6.2) \qquad \begin{aligned} a_2 \sinh \bar{x} + a_3 \cosh \bar{x} &= - \epsilon_1 a_1 \\ b_2 \sinh \bar{x} + b_3 \cosh \bar{x} &= - \epsilon_1 b_1 \end{aligned} .$$

If there exists an $\bar{x}$ satisfying (6.2) then, since the equations are consistent but not proportional, $|a_2 b_3| \neq 0$ and solution of (6.2) yields

$$(6.3) \qquad \sinh \bar{x} = \epsilon_1 \frac{|a_3 b_1|}{|a_2 b_3|}, \quad \cosh \bar{x} = \epsilon_1 \frac{|a_1 b_2|}{|a_2 b_3|} .$$

Substitution of (6.3) into the relation $\cosh^2 \bar{x} - \sinh^2 \bar{x} = 1$ yields (6.1) immediately.

If, conversely, $|a_2 b_3| \neq 0$ and (6.1) hold, then (6.1) can be in the form

$$\frac{|a_1 b_2|^2}{|a_2 b_3|^2} - \frac{|a_3 b_1|^2}{|a_2 b_3|^2} = 1.$$

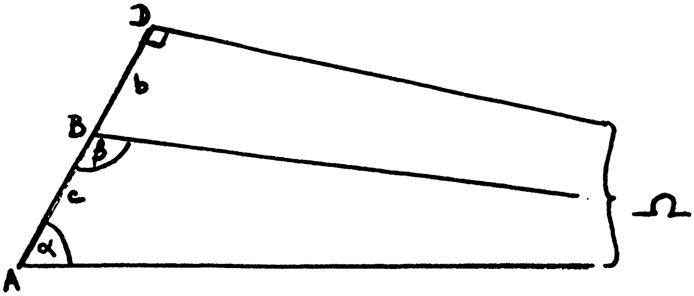
It follows that there is an $\bar{x}$ determined by (6.3) if $\epsilon_1$ is chosen so that $\cosh \bar{x} > 0$. This $\bar{x}$ then satisfies (6.2) and the lines are parallel. This completes the proof that (6.1) is a necessary and sufficient condition for parallelism and shows in addition that there is parallelism with the $x$-axis if and only if $|a_2 b_3| = 0$.

As a corollary to Theorems 4.1 and 6.1 we obtain at once by exclusion

**Theorem 6.2.** *A necessary and sufficient condition that two distinct lines* **a** *and* **b** *be nonintersecting is that* $|a_1b_2|^2 - |a_2b_3|^2 - |a_3b_1|^2 < 0$.

## 7. *Angles of Ω Triangles*

Consider the figure consisting of two parallels $A\Omega$ and $B\Omega$ and the transversal $AB$.




The relation between the interior angles $\alpha$ and $\beta$ and the length $c = AB$ is obtained as follows. For definiteness, it may be assumed that $\alpha < \pi/2$. Let $D\Omega$ be the common parallel to $A\Omega$ and $B\Omega$ which is perpendicular to $AB$. Then $AD = a$ is the distance of parallelism for $\alpha$ and $BD = b$ is the distance of parallelism for $\pi - \beta$. Then, it is known [3, p. 148],

$$\cos\alpha = \tanh a$$

$$\cos\beta = -\cos(\pi - \beta) = -\tanh b$$

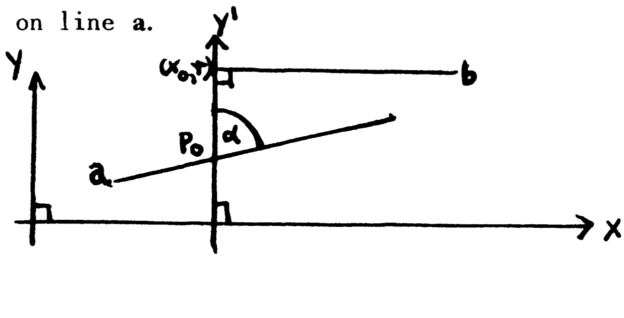
But, $c = a - b$, where it is understood $b < 0$ if $\beta < \pi/2$, whence:

$$\tanh c = \frac{\tanh a - \tanh b}{1 - \tanh a \tanh b} = \frac{\cos\alpha + \cos\beta}{1 + \cos\alpha\cos\beta}, \tag{7.1}$$

which is the desired relation. Note that as $c$ approaches zero, it follows from (7.1) that $\alpha + \beta$ approaches $\pi$, the euclidean relation.

## 8. *Angles*

Let $P_0(x_0, y_0)$ lie on line **a**.



Assuming $a_1 \neq 0$, let us determine the positive angle from **a** to the line $x = x_0$, defining " positive angle" in the usual counterclockwise sense. Consider a new coordinate system which retains the same $x$-axis but elects $x = x_0$ for the $y$-axis. The new $x'y'$ system is related to the original system by:

$$(8.1) \qquad x' = x - x_0, \quad y' = y.$$

In this new system, $P_0$ has coordinates $(0, y_0)$ and line **a** has for its equation:

$$(8.2) \qquad A_1 \tanh y' + A_2 \sinh x' + A_3 \cosh x' = 0, \text{ where}$$

$$(8.3) \quad A_1 = a_1, \; A_2 = a_3 \sinh x_0 + a_2 \cosh x_0, \; A_3 = a_3 \cosh x_0 + a_2 \sinh x_0.$$

Moreover, since $P_0(0, y_0)$ satisfies (8.2), $A_3 = -A_1 \tanh y_0$, so **a** takes the form:

$$(8.4) \qquad A_1 \tanh y' + A_2 \sinh x' - A_1 \tanh y_0 \cosh x' = 0.$$

Now let **b** be parallel to **a** to the right and perpendicular to the $y'$-axis. But the equation of **b** is [3, p. 145]

$$\tanh y' = \tanh r \cosh x',$$

where $r$ is the $y'$ intercept. Application of the parallelism condition (Theorem 6.1), to **a** and **b** yields:

$$\sinh (r - y_0) = \pm \frac{A_2}{A_1} \cosh y_0.$$

If $\alpha$ is acute, the right hand parallel to **a** meets the $y'$-axis above $P_0$, so $(r - y_0)$ is the distance of parallelism for $\alpha$. If $\alpha$ is obtuse, **b** meets the $y'$-axis below $P_0$ [3, p.77], so $r < y_0$ and $(r - y_0)$ is the distance of parallelism for $\alpha$. But $\alpha$ is acute or obtuse according as $\frac{dy'}{dx'} > 0$, or $<0$ for **a** at $P_0$. (Note that the derivative here is to be interpreted only as an instantaneous rate of change of one variable with respect to another.)

Now for **a**, $\frac{dy'}{dx'} = -\frac{A_2}{A_1} \cosh^2 y_0$, at $P_0$ so if $\alpha$ is acute, $\frac{-A_2}{A_1} > 0$, hence: $\sinh (r - y_0) = -\frac{A_2}{A_1} \cosh y_0$. If $\alpha$ is obtuse, $-\frac{A_2}{A_1} < 0$ and: $\sinh (r - y_0) = -\frac{A_2}{A_1} \cosh y_0$.

Also, since $(r - y_0)$ is the distance of parallelism for $\alpha$, in every case the final result [3, p.151] is

$$\cot \alpha = -\frac{A_2}{A_1} \cosh y_0. \tag{8.5}$$

Note that this formula applies even when $A_1$ or $A_2$, but not both, vanish.

Suppose now that **a** and **b** are any two lines through $P_0$, respectively forming positive angles $\alpha$ and $\beta$ with the $y'$-axis. Then the directed angle $\theta$ from **a** to **b** is given by $\theta = \alpha - \beta$, whence:

$$\cot \theta = \frac{A_1B_1 + A_2B_2 \cosh^2 y_0}{(A_2B_1 - B_2A_1) \cosh y_0}$$

By use of (8.3) and the appropriate formulas for **b**, this last result may be expressed in the original coordinate system by:

$$\cot \theta = \frac{a_1b_1 + (a_3 \tanh x_0 + a_2)(b_3 \tanh x_0 + b_2) \cosh^2 x_0 \cosh^2 y_0.}{(-|a_1b_2| + |a_3b_1| \tanh x_0) \cosh x_0 \cosh y_0} \tag{8.6}$$

Substitution of (4.1) and (4.3) into (8.6) yields, after some unexciting calculation,

$$\cot \theta = \frac{-\epsilon(a_1b_1 + a_2b_2 - a_3b_3)}{\sqrt{|a_1b_2|^2 - |a_2b_3|^2 - |a_3b_1|^2}} \tag{8.7}$$

Note that since $\epsilon = \pm 1$, it may be written in either the numerator or denominator at pleasure.

It will be convenient to introduce the following notation. Let

$$[a/b] = a_1b_1 + a_2b_2 - a_3b_3. \tag{8.8}$$

Since Lemma 5.1 may be written $|a_1b_2|^2 - |a_2b_3|^2 - |a_3b_1|^2 = [a/a] \cdot [b/b] - [a/b]^2$, formula (8.7) becomes :

$$\cot \theta = \frac{+\delta[a/b]}{\sqrt{[a/a] \cdot [b/b] - [a/b]^2}} \tag{8.9}$$

where $\delta$ is $\pm 1$ and its sign is opposite to that of $|a_1b_2|$. This formula (8.9) is valid no matter which directed angle from **a** to **b** is denoted by $\theta$. If it is agreed that $\theta$ denote the smallest positive angle from **a** to **b**, then $0 \leq \theta < \pi$, whence

$$(8.10)\qquad \sin\theta = \frac{\sqrt{[a/a]\cdot[b/b]-[a/b]^2}}{\sqrt{[a/a][b/b]}}, \cos\theta = \frac{\delta[a/b]}{\sqrt{[a/a][b/b]}}.$$

It may be noted that in the new notation, a necessary and sufficient condition that (3.1) represent a line is the $[a/a] > 0$.

## 9. *Perpendicularity*

**Lemma 9.1.** *If* $[a/b] = 0$, *lines* **a** *and* **b** *intersect.*

For, by Theorem 4.1, the condition for intersection is:

$|a_1b_2|^2 - |a_2b_3|^2 - |a_3b_1|^2 > 0$. Since $a_3b_3 = a_1b_1 + a_2b_2$, we have by Lemma 5.1: $|a_1b_2|^2 - |a_2b_3|^2 - |a_3b_1|^2 = (a_1^2 + a_2^2 - a_3^2)(b_1^2 + b_2^2 - b_3^2) > 0$, since each factor is positive by Theorem 3.1.

**Theorem 9.1.** *A necessary and sufficient condition that* **a** *and* **b** *be perpendicular is that* $[a/b] = 0$. (The proof follows from Lemma 9.1 and the second equation of (8.10) ).

**Theorem 9.2.** *The line through* $(x_0, y_0)$ *and perpendicular to* **a** *has the equation:*

$$\begin{vmatrix} \tanh y & \sinh x & \cosh x \\ \tanh y_0 & \sinh x_0 & \cosh x_0 \\ a_1 & a_2 & -a_3 \end{vmatrix} = 0.$$

For, if the line is **b**, since it passes through $(x_0, y_0)$, $b_1 \tanh y_0 + b_2 \sinh x_0 + b_3 \cosh x_0 = 0$. By Theorem 9.1, $a_1b_1 + a_2b_2 - a_3b_3 = 0$, whence elimination of $b_1$, $b_2$, $b_3$ yields the equation. Note that the last two rows of the determinant are not proportional since, $a_3^2 - a_2^2 < a_1^2$, and, $\cosh^2 x_0 - \sinh^2 x_0 > \tanh^2 y_0$.

## 10. *Non-Intersection*

Let the lines **a** and **b** have a common perpendicular, given by **c**. Then, $a_1c_1 + a_2c_2 - a_3c_3 = 0$, $b_1c_1 + b_2c_2 - b_3c_3 = 0$. Since **a** and **b** are distinct, $c_1$, $c_2$ and $c_3$ can be chosen to be, respectively, $|a_2b_3|$, $|a_3b_1|$, $-|a_1b_2|$. These coefficients actually determine a line if and only if: $c_3^2 - c_2^2 < c_1^2$, i.e., if and only if: $|a_2b_1|^2 - |a_1b_3|^2 < |a_3b_2|^2$. Hence:

**Theorem 10.1.** *A necessary and sufficient condition that* **a** *and* **b** *be non-intersecting is that:* $|a_1b_2|^2 - |a_2b_3|^2 - |a_3b_1|^2 < 0$. *Their common perpendicular is given by:* $|a_2b_3| \tanh y + |a_3b_1| \sinh x - |a_1b_2| \cosh x = 0$.

(Note, as a partial summary, the condition that two lines be intersecting, parallel or non-intersecting, respectively, is that:

$$[a/a][b/b] - [a/b]^2 > 0, = 0, < 0.)$$

## 11. *Families of lines*

Consider two distinct lines **a** and **b** and form the linear combination: (11.1) $k(a_1\tanh y+a_2\sinh x+a_3\cosh x)+h(b_1\tanh y+b_2\sinh x+b_3\cosh x)=0$, $k^2+h^2 \neq 0$. Since this is of form (3.1), it is either a straight line or has no locus. If **a** and **b** are intersecting, the point of intersection satisfies (11.1) and for suitable $k$ and $h$, (11.1) may be made to pass through an arbitrary point. If **a** and **b** are non-intersecting, let their common perpendicular be the line **c**. By Theorem 9.2, $a_1c_1 + a_2c_2 - a_3c_3 = b_1c_1 + b_2c_2 - b_3c_3 = 0$. Also by Theorem 9.2, (11.1) is perpendicular to **c** for: $(ka_1 + hb_1)c_1 + (ka_2 + hb_2)c_2 - (ka_3 + hb_3)c_3 = 0$. When **a** and **b** are non-intersecting, it follows that (11.1) contains all lines having the same common perpendicular. For some $k$ and $h$, however, (11.1) may have no locus. If **a** and **b** are parallel, then $[a/a][b/b] - [a/b] = 0$, as noted above. Consider any two distinct lines of family (11.1), $k_1a + h_1b = 0$ and $k_2a + h_2b = 0$. If $k_1h_2 - k_2h_1$ is denoted by $R$, it may be verified directly that $[k_1a + h_1 b/k_1 a + h_1b][k_2a + h_2 b/k_2 a + h_2b] - [k_1a + h_1 b/k_2 a + h_2b] = R^2\{[a/a][b/b] - [a/b]^2\} = 0$, so that if **a** and **b** are parallel all members of (11.1) are parallel to each other, and in particular are parallel to **a** and **b**. Actually, all members of (11.1) must be parallel to **a** and **b** in their direction of parallelism, for while there is a line **c** parallel to **a** and **b** in the direction opposite to that of parallelism, there exist no lines parallel to **a**, **b**, and **c**, whence **c** cannot be a member of family (11.1). Thus if **a** and **b** are parallel, (11.1) is the family of all lines parallel to them in their direction of parallelism.

If three or more lines are parallel in the same sense, they are said to be concurrent at an ideal point, while if they have a common perpendicular, they are said to be concurrent in an ultra-ideal point [3, p.77, p.85]. If the term "concurrent" is extended to include these types, the relations above result in:

**Theorem 11.1.** *The family of lines which are linear combinations of distinct lines* **a** *and* **b** *consists of all lines "concurrent" with* **a** *and* **b**.

Three lines, $a$, $b$, $c$ are linearly dependent if there exist constants $k$, $h$, $m$, not all zero, such that

(11.2)
$$k(a_1\tanh y + a_2\sinh x + a_3\cosh x) +$$
$$h(b_1\tanh y + b_2\sinh x + b_3\cosh x) + m(c_1\tanh y + c_2\sinh x + c_3\cosh x) \equiv 0.$$

That is, one of the lines is a linear combination of the other two. If any two of the lines are identical, they are trivially "concurrent" and also linearly dependent. Otherwise, Theorem 11.1 applies and we have:

**Theorem 11.2.** *A necessary and sufficient condition that three lines be "concurrent" is that they be linearly dependent.*

Also, since the only way identity (11.2) can hold is to have the coefficients of each variable term zero, it follows that:

**Theorem 11.3.** *A necessary and sufficient condition that lines* **a**, **b**, *and* **c** *be "concurrent" is:*

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

## 12. *Distance from a point to a line*

Let $P_0(x_0, y_0)$ be an arbitrary point and **a** an arbitrary line. The line **b** through $P_0$ which is perpendicular to **a** has as its equation (Theorem 9.2)

(12.1)
$$\begin{vmatrix} \sinh x_0 & \cosh x_0 \\ a_2 & -a_3 \end{vmatrix} \tanh y + \begin{vmatrix} \cosh x_0 & \tanh y_0 \\ -a_3 & a_1 \end{vmatrix} \sinh x + \begin{vmatrix} \tanh y_0 & \sinh x_0 \\ a_1 & a_2 \end{vmatrix} \cosh x = 0.$$

For later computation, it will be convenient to note that:

$$
\begin{aligned}
|a_1b_3| &= -|a_3b_1| = a_2r - d^2 \sinh x_0, \\
(12.2) \qquad |a_3b_2| &= -|a_2b_3| = a_1r - d^2 \tanh y_0 \\
|a_2b_1| &= -|a_1b_2| = -a_3r - d^2 \cosh x_0
\end{aligned}
$$

where, $r = a_1\tanh y_0 + a_2\sinh x_0 + a_3\cosh x_0$, and $d = \sqrt{[a/a]} > 0$. The point of intersection $(x_1, y_1)$ of **a** and **b** may be determined directly from (4.1). Since $(x_1, y_1)$ is known to exist, it follows that $|a_1b_2|^2 > 0$ for all points $(x_0, y_0)$. Using (12.2), $|a_1b_2|$ is a continuous function of $(x_0, y_0)$ and is positive when $(x_0, y_0)$ is on **a**, since there $r = 0$. Hence for (4 1), $\epsilon = +1$. By a little manupulation, it can shown that:

$$|a_1b_2|^2 - |a_3b_1|^2 = r^2(a_1^2 + d^2) - 2a_1r\,d^2\tanh y_0 + d^4.$$

Let $\zeta = +\sqrt{|a_1b_2|^2 - |a_3b_1|^2}$. By (4.1) it results that:

$$\tanh y_1 = \frac{d^2\tanh y_0 - a_1r}{\zeta}, \quad \sinh x_1 = \frac{d^2\sinh x_0 - a_2r}{\zeta}, \quad \cosh x_1 = \frac{d^2\cosh x_0 + a_3r}{\zeta}.$$

By (4.3)

$$\sinh y_1 = \frac{(d^2\tanh y_0 - a_1r)\cosh y_0}{d\sqrt{r^2\cosh^2 y_0 + d^2}}, \quad \cosh y_1 = \frac{\zeta \cosh y_0}{d\sqrt{r^2\cosh^2 y_0 + d^2}}$$

Now, the distance $s$ from $(x_0, y_0)$ to **a** is the distance between $(x_0, y_0)$ and $(x_1, y_1)$. Hence, by substitution of these latter formulas, it may be shown that (2.1) reduces to:

$$\cosh s = \sqrt{1 + \frac{r^2}{d^2}\cosh^2 y_0}\ .$$

Letting $s$ be only positive permits this last relation to be expressed by by:

$$\sinh s = \frac{|r|\cosh y_0}{d}, \quad \text{or:}$$

$$(12.4) \quad \sinh s = \frac{|a_1\tanh y_0 + a_2\sinh x_0 + a_3\cosh x_0|\cosh y_0}{\sqrt{[a/a]}}.$$

If the absolute values are deleted in (12.3), the resulting formula may be said to give a directed distance $s$, positive for points on one side of **a**, negative for those on the other. This can be shown by recognizing first that sinh $s$ is a continuous function of $x_0$ and $y_0$ which is zero only on line **a**. If it is agreed to choose the equation for **a** with $a_3 < 0$, then the following formula is unique:

$$\text{(12.4)} \quad \sinh s = \frac{(a_1 \tanh y_0 + a_2 \sinh x_0 + a_3 \cosh x_0)\cosh y_0}{\sqrt{[a/a]}} .$$

(This covers all cases except where $a_3 = 0$, when the line goes through the origin, a case which is handled easily. Since (12.4) gives a negative answer for (0,0), it follows that it gives a negative answer for any point on the same side of **a** as the origin, and positive answers for all points on the side opposite from the origin.

The equation for **a** is said to be in normal form if $a_3 \leq 0$ and $\sqrt{[a/a]} = 1$. A line **a** may be put in normal form by dividing by $\pm\sqrt{[a/a]}$ and choosing the sign opposite to $a_3$. Hence:

**Theorem 12.1.** *If the equation of a line* **a** *is written in normal form, the hyperbolic sine of the directed distance from* $(x_0, y_0)$ *to* **a** *is obtained by substituting* $(x_0, y_0)$ *in the normal form of* **a** *and multiplying by cosh* $y_0$.

## 13. *Analytic proofs of geometric theorems*

With the machinery developed, many complicated synthetic proofs of elementary theorem are easily proved analytically.

**Theorem 13.1.** *The altitudes of a triangle are "concurrent".*

To prove this let the sides of the triangle be **a**, **b**, and **c**. The altitude on **a** is a linear combination of **b** and **c** which is perpendicular to **a**. Hence we have:

$$(kb_1 + hc_1)\tanh y + (kb_2 + hc_2)\sinh x + (kb_3 + hc_3)\cosh x = 0, \text{ and,}$$

$$a_1(kb_1 + hc_1) + a_2(kb_2 + hc_2) - a_3(kb_3 + hc_3) = 0, \text{ or}$$

$$k[a/b] + h[a/c] = 0,$$

whence we may choose $k = [a/c]$, $h = -[a/b]$. Notice $k$ and $h$ are not both 0, since **b** and **c** cannot both be perpendicular to **a**. Hence the altitude on **a** has for it equation:

$$\text{(13.1)} \quad \begin{aligned} &[a/c](b_1 \tanh y + b_2 \sinh x + b_3 \cosh x) - \\ &[a/b](c_1 \tanh y + c_2 \sinh x + c_3 \cosh x) = 0. \end{aligned}$$

Similarly, the perpendiculars on sides **b** and **c** are given by:

(13.2) $$[a/b](c_1 \tanh y + c_2 \sinh x + c_3 \cosh x) - [b/c](a_1 \tanh y + a_2 \sinh x + a_3 \cosh x) = 0,$$

(13.3) $$[b/c](a_1 \tanh y + a_2 \sinh x + a_3 \cosh x) - [a/c](b_1 \tanh y + b_2 \sinh x + b_3 \cosh x) = 0.$$

But (13.1),(13.2), and (13.3) are linearly dependent with constants (1, 1, 1). Hence, the altitudes are "concurrent" (Theorem 11.2).

**Theorem 13.2.** *The three internal bisectors of the angles of a triangle are "concurrent". The internal bisector at any vertex is "concurrent" with the external bisectors at the other two vertices.*

Using Theorem 12.1 the bisectors may be found as usual in analytic geometry and the proof follows by Theorem 11.2.

**Theorem 13.3** *The perpendicular bisectors of the sides of a triangle are "concurrent".*

The proof follows by using formula (2.2) and then Theorem 11.2.

**Theorem 13.4** *The medians of a triangle are "concurrent".*

Let the sides of the triangle be **a, b, c**. Let the vertices of the triangle be $A$, $B$, $C$, respectively opposite **a, b, c**. Let $(D/d)$ denote the result of substituting point $D$ into the equation of line **d**. Now, the linear combination:

(13.4) $$k(a_1 \tanh y + a_2 \sinh x + a_2 \cosh x) + h(b_1 \tanh y + b_2 \sinh x + b_3 \cosh x) = 0,$$

will be a median through $C$ if the directed distances from it to $A$ and $B$ are equal and opposite in sign. After reducing (13.4) to normal form, the technique explained yields

(13.5) $\{k(A/a) + h(A/b)\}\cosh y_1 + \{k(B/a) + h(B/b)\}\cosh y_2 = 0$, or

(13.6) $k(A/a)\cosh y_1 + h(B/b)\cosh y_2 = 0$, since $(B/a) = (A/b) = 0$. Thus it is possible to take $k = (B/b)\cosh y_2$, $h = -(A/a)\cosh y_1$. Deducing the equations of the other medians and applying Theorem 11.2, the theorem is readily proved.

## 14. *Embedding in the projective plane*

Much of the simplicity of the results deduced depends on the fact that equation (3.1) is linear in the coefficients. It seems natural to inquire whether it is possible to introduce a coordinate system in which the equation of a line is also linear in the coordinates of the moving point. To this end, introduce new coordinates $(u,v)$ defined by

$$(14.1) \qquad u = \frac{\tanh y}{\cosh x}, \quad v = \tanh x.$$

Equation (2.5) reduces at once to: $a_1u + a_2v + a_3 = 0$, which has the desired form. Equations (14.1), solved for $x$ and $y$, take the form:

$$(14.2) \qquad x = \tanh^{-1} v, y = \tanh^{-1} \frac{u}{\sqrt{1 - v^2}}$$

By (14.1), to each point $(x,y)$ corresponds a unique pair $(u,v)$ while by (14.2), a given pair $(u,v)$ corresponds to a point $(x,y)$ if and only if

$$v^2 < 1, \text{ and, } \quad \frac{u^2}{1 - v^2} < 1,$$

which are both valid if and only if $u^2 + v^2 < 1$. This new coordinate system therefore represents the points of the hyperbolic plane by number couples $(u,v)$ satisfying the condition: $u^2 + v^2 < 1$, and the lines of the hyperbolic plane are the sets of points represented by pairs $(u,v)$ satisfying the above condition and a linear equation:

$$a_1u + a_2v + a_3 = 0.$$

Since the number couples $(u,v)$ with $u^2 + v^2 < 1$ can be thought of as the ordinary cartesian coordinates of the points inside the unit circle in the euclidean plane, this new system sets up a 1-1 correspondence of all the points of the hyperbolic plane with the points inside the unit circle in the euclidean plane, This is perhaps the simplest model of hyperbolic geometry and is often described by saying that the hyperbolic plane can be embedded in the euclidean plane.

It is convenient to introduce homogeneous coordinates $(u_1,u_2,u_3)$ defined by:

$$u = \frac{u_1}{u_3}, \quad v = \frac{u_2}{u_3},$$

which allow us to consider the extended euclidean plane with its ideal points. The points of hyperbolic geometry are denoted by the triples $(u_1,u_2,u_3)$ with $[u/u] < 0$. This point of view toward hyperbolic geometry proves to be most revealing. For example, the equation: $a_1u_1 + a_2u_2 + a_3u_3 = 0$, always represents a line in the extended euclidean plane whose distance from the origin, if it exists, is

$\frac{|a_3|}{\sqrt{a_1^2 + a_2^2}}$. In order for the line to have a segment interior to the unit circle, this distance must be less than one unit, that is, $\frac{a_3^2}{a_1^2+a_2^2}<1$.

Thus, $a_1u_1 + a_2u_2 + a_3u_3 = 0$ represents a line in the hyperbolic plane if and only if $[a/a] > 0$. Otherwise the equation has no locus. This sheds light on the existence condition of Theorem 3.1.

Consider now two arbitrary distinct lines **a** and **b** with equations

$$a_1u_1 + a_2u_2 + a_3u_3 = 0,$$

$$b_1u_1 + b_2u_2 + b_3u_3 = 0.$$

The corresponding lines in the extended euclidean plane always have a point in common, namely the point $(|a_2b_3|, |a_3b_1|, |a_1b_2|)$. This is a hyperbolic point if and only if: $|a_2b_3|^2 + |a_3b_1|^2 - |a_1b_2|^2 < 0$. But this is the condition for intersection stated in Theorem 4.1. By Theorem 6.1, **a** and **b** are parallel if and only if: $|a_2b_3|^2 + |a_3b_1|^2 - |a_1b_2|^2 = 0$, i.e., if and only if the euclidean intersection is on the unit circle. It is possible therefore to identify the ideal points of hyperbolic geometry with the euclidean points on the unit circle. Finally, **a** and **b** are non-intersecting if their euclidean intersection is outside the unit circle, i.e., if: $|a_2b_3|^2 + |a_3b_1|^2 - |a_1b_2|^2 > 0$. which is the condition of Theorem 10.1. Hence, the ultra-ideal point for the lines **a** and **b** may be identified with this euclidean intersection. Hence, the ideal and ultra-ideal points which were originally conveniences of terminology may now be viewed in a concrete way and serve to clarify the facts of hyperbolic geometry.

The parallelism between the analytic description of point and line is now quite striking. A triple of numbers **a** may denote either a point or a line, that is, may be either the coordinates of a point or the coefficients in the equation of a line. If **a** and **b** denote hyperbolic (or euclidean) lines, then the triple $(|a_2b_3|, |a_3b_1|, |a_1b_2|)$ are the coordinates of the euclidean intersection which may or may not be a hyperbolic point. Similarly, if **a** and **b** denote hyperbolic points, then the triple $(|a_2b_3|, |a_3b_1|, |a_1b_2|)$ are the coeffieients of the hyperbolic line joining them, for both **a** and **b** satisfy the equation: $x_1|a_2b_3| + x_2|a_3b_1| + x_3|a_1b_2| = 0$.

It should be noted that for a triple **a** to represent a hyperbolic point it is necessary that $[a/a] < 0$, while for it to represent a hyperbolic line, it is necessary that $[a/a] > 0$.

The embedding of the hyperbolic plane in the extended euclidean (or projective) plane suggests the possibility of interpreting other concepts and formulas of hyperbolic geometry in terms of the surrounding projective geometry. For example, the pole of a hyperbolic line **a** with respect to the unit circle is the (ultra-ideal) point $(a_1, a_2, -a_3)$. The condition of perpendicularity of **a** and **b**, namely $[a/b] = 0$, is therefore precisely the condition that **b** pass through the pole of **a**. Thus, two hyperbolic lines are perpendicular if the corresponding euclidean lines are conjugate, i.e., each passes through the pole of the other. Hence, perpendicularity in hyperbolic geometry is related to the projective theory of poles and polars.

Since cross ratio is the fundamental invariant of projective geometry, it is natural to suspect that some of the analytic formulas that have been developed for hyperbolic geometry may be expressible in terms of cross ratios. Let $R$ and $S$, with coordinates $r$ and $s$ be two hyperbolic points. The euclidean line joining them meets the unit circle in points $P^*$ and $Q^*$. The cross ratio of these four points, $(RS,P^*Q^*)$ may be readily computed and is found to be

$$(14.3) \qquad (RS,P^*Q^*) = \frac{-[r/s] + \sqrt{[r/s]^2 - [r/r][s/s]}}{-[r/s] - \sqrt{[r/s]^2 - [r/r][s/s]}}.$$

The formula for the distance $d$ between $r$ and $s$, as given by (2.1) turns out to be:

$$(14.4) \quad d = \frac{1}{2} \ln \left\{ \frac{-[r/s] + \sqrt{[r/s]^2 - [r/r][s/s]}}{-[r/s] + \sqrt{[r/s]^2 - [r/r][s/s]}} \right\} = \frac{1}{2} \ln (RS,P^*Q^*).$$

This shows how distance in hyperbolic geometry may be interpreted in terms of cross ratio.

Similarly, the angle between intersecting hyperbolic lines can be expressed in terms of the cross ratio of the two lines and the tangents to the unit circle through their intersection. Of course, all the formulas developed for the hyperbolic coordinates $(x,y)$ can be readily expressed in the homogeneous coordinates.

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University of Maryland

# MISCELLANEOUS NOTES

*Edited by*

Charles K. Robbins

Articles intended for this Department should be sent to Charles K. Robbins, Department of Mathematics, Purdue University, Lafayette, Indiana.

## POLAR SYMMETRY

R. Lariviere

Any polar point $(r,\theta)$ has generalized coordinates $(-1)^n r,\ \theta + n\pi$. It has been shown [1,2] that to determine all the points of intersection of two polar curves, except the pole, it is only necessary to solve simultaneously the equations of the two curves, having previously expressed one of them in generalized coordinates for which $n=0, \pm 1, \cdots$

The coordinates may be used advantageously in another area - that of polar symmetry. The dissatisfaction of the student who finds that the common tests fail to reveal an existing symmetry may be allayed by showing him the tests with generalized coordinates as in the following example.

Test the curve

$$r = 1 - 2\sin(\theta/3) \tag{1}$$

for symmetry about the $90°$ axis.

The simple substitutions of $\pi-\theta$ for $\theta$, $-r$ for $r$ and $-\theta$ for $\theta$ yield

$$r = 1 - 2\sin(\pi - \theta)/3, \tag{2}$$

$$-r = 1 - 2\sin(-\theta/3), \tag{3}$$

neither of which reduces to the original equation.

If, however, we write equation (1) in generalized coordinates $(-1)^n r = 1 - 2\sin(\theta + n\pi)/3$ we obtain (2) when $n=2$ and (3) when $n=3$:

$$(-1)^2 r = 1 - 2\sin[3\pi - (\pi - \theta)]/3 = 1 - 2\sin(\pi - \theta)/3;$$

$$(-1)^3 r = 1 - 2\sin(3\pi + \theta)/3 = 1 + 2\sin\theta/3 = 1 - 2\sin(-\theta/3).$$


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University of Illinois in Chicago.

## HOW TO DERIVE THE FORMULA "$H_r^n = C_r^{n+r-1}$"

Chung Lie Wang

*Introduction*

In studying permutations and combinations we have become acquainted with the formula

$$H_r^n = C_r^{n+r-1}$$

for the combinations with repetitions. But the deriving of it was simply included in the statement:

> *The number of the r-combinations with repetitions of n different things is the same as the number of the r-combinations without repetitions of n+r-1 different things, namely,* $C_r^{n+r-1}$.

It doesn't seem logical enough for those who have particular interest in this subject and may be in want of a more solid proof.

**The Basic Formulas**

Two well known fundamental formulas may be quoted as follows:

(I) $$C_r^n = C_r^{n-1} + C_{r-1}^{n-1}$$

(II) $$C_r^{m+n} = C_r^m + C_{r-1}^m \cdot C_1^n + C_{r-2}^m \cdot C_2^n + \dots + C_1^m \cdot C_{r-1}^n + C_r^n .$$

By using the formula (I) repeatedly we get its generalization.

Since $$C_r^{n-1} = C_r^{n-2} + C_{r-1}^{n-2} \dots \qquad (1)$$

$$C_{r-1}^{n-1} = C_{r-1}^{n-2} + C_{r-2}^{n-2} \dots \qquad (2)$$

(1) + (2): $$C_r^n = C_r^{n-2} + 2C_{r-1}^{n-2} + C_{r-2}^{n-2},$$

or $$C_r^n = C_r^{n-2} + C_1^2 \cdot C_{r-1}^{n-2} + C_{r-2}^{n-2} \qquad (3)$$

Similarly $$C_r^n = C_r^{n-3} + C_1^3 \cdot C_{r-1}^{n-3} + C_2^3 \cdot C_{r-2}^{n-3} + C_{r-3}^{n-3} \qquad (4)$$

Therefore, proceeding as above and according to mathematical induction, we have

(III) $$C_r^n = C_r^{n-r+1} + C_1^{r-1} \cdot C_{r-1}^{n-r+1} + C_2^{r-1} \cdot C_{r-2}^{n-r+1} + \dots + C_1^{n-r+1} .$$

**Proof of the Formula**

We shall take account of all the $r$-combinations with repetitions of $n$ different letters, once each, if we classify them as follows.

They consist of

(A) The combinations in which no repetition is allowed. (The number of these combinations is $C_r^n$.)

(B) The combinations in which repetitions are allowed. These are classified as follows, and formula (III) is introduced.

(i) The combinations which contain $r$-1 letters of $n$ letters and one letter of the same $r$-1 letters. As we can choose

(a) the $r$-1 letters in $C_{r-1}^n$ ways,

(b) the one letter in $C_1^{r-1}$ ways,

the number of combinations of this kind is $C_{r-1}^n \cdot C_1^{r-1}$.

(ii) The combinations which contain $r$-2 letters of $n$ letters and two letters of the same $r$-2 letters. As we can choose

(a) the $r$-2 letters in $C_{r-2}^n$ ways.

(b) the two letters in two cases. They consist of

(1) two different letters in $C_2^{r-2}$ ways,

(2) two same letters in $C_1^{r-2}$ ways.

The number is $C_2^{r-2} + C_1^{r-2} = C_2^{r-1}$.

The number of combinations of this kind is $C_{r-2}^n \cdot C_2^{r-1}$.

(iii) The combinations which contain $r$-3 letters of $n$ letters and three letters of the same $r$-3 letters. As we can choose

(a) the $r$-3 letters in $C_{r-3}^n$ ways,

(b) the three letters in three cases. They consist of

(1) three different letters in $C_3^{r-3}$ ways,

(2) two letters which may be the same in $C_1^2 \cdot C_2^{r-3}$ ways,

(3) all three letters which are the same in $C_1^{r-3}$ ways.

The number is $C_3^{r-3} + C_1^2 \cdot C_2^{r-3} + C_1^{r-3} = C_3^{r-1}$.

The number of combinations of this kind is $C_{r-3}^n \cdot C_3^{r-1}$.

And so on, until last of all we reach the combinations in which $r$ letters are the same. That is, as we can choose the one letter in $C_1^n$ ways, the number of combinations of this kind is

$$C_1^n = C_1^n \cdot C_{r-1}^{r-1} \quad (\text{since } C_{r-1}^{r-1} = 1)$$

Therefore, we get

$$H_r^n = C_r^n + C_{r-1}^n \cdot C_1^{r-1} + C_{r-2}^n \cdot C_2^{r-1} + \ldots + C_1^n \cdot C_{r-1}^{r-1}.$$

For the sake of making this equation of the same type as formula (II) we add $C_r^{r-1} = 0$ to both sides. Then we have

$$H_r^n = C_r^n + C_{r-1}^n \cdot C_1^{r-1} + C_{r-2}^n \cdot C_2^{r-1} + \ldots + C_1^n \cdot C_{r-1}^{r-1} + C_r^{r-1}.$$

Since

$$C_r^n + C_{r-1}^n \cdot C_1^{r-1} + C_{r-2}^n \cdot C_2^{r-1} + \ldots + C_1^n \cdot C_{r-1}^{r-1} + C_r^{r-1} = C_r^{n+r-1}$$

therefore,

$$H_r^n = C_r^{n+r-1}$$

Ordnance Engineering College
Taipei, Taiwan

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## ON SOME MAXIMUM-MINIMUM PROBLEMS

R. K. Morley

This note deals with making up certain polynomials for illustrating the application of derivatives to finding maximum and minimum points and the points of inflection.

Suppose $y = f(x)$ to be a rational integral polynomial with rational coefficients. Let us assume $f'(x) = ax^{n+2} + bx^{n+1} + cx^n$,[1] $n$ an integer $\geq 1$, $a$, $b$, $c$ rational $\neq 0$. This will offer in one example ample illustrations of the points in question and will require (except for zero roots) solving no equation beyond a quadratic. The question we discuss is whether it is possible to have rational roots in both

(1) $$f'(x) = 0$$

and

(2) $$f''(x) = 0$$

and if so to find values of $a$, $b$, and $c$ that will make all these roots rational. $f(x)$ may then be found by integrating $f'(x)$.

The two cases one is most likely to use are $f(x)$ a quartic, $n = 1$, and $f(x)$ a quintic, $n = 2$, but we will first consider the question with general $n$ and discuss those special cases by examples later.

For the roots of (1) to be rational we must have $b^2 - 4ac = p^2$,

$p$ rational, or

(3) $$ac = \frac{b^2 - p^2}{4}$$

Now $f''(x) = (n+2)ax^{n+1} + (n+1)bx^n + ncx^{n-1}$ and for the roots of (2) to be rational we must have $(n+1)^2b^2 - 4n(n+2)ac = q^2$, $q$ rational, or

(4) $$ac = \frac{(n+1)^2b^2 - q^2}{4n(n+2)}$$

From (3) and (4)

(5) $$q^2 - b^2 = n(n+2)p^2$$

or

(6) $$(q+b)(q-b) = n(n+2)p^2$$

This can be satisfied in various ways by setting the factors of the left hand member equal to factors of the right hand member. We must be careful to avoid making $b = p$, since from (3) $ac = 0$.

For instance, let $p = vw$, $v$ and $w$ rational, and let $q + b = v^2$, $q - b = n(n+2)w^2$. (6) is satisfied. We have

$$b = \frac{v^2 + n(n+2)w^2}{2}$$

and

(7) $$p = \frac{v^2 - n(n+2)w^2}{2}$$

$p$ and $q$ are rational.

From (3) $$ac = \frac{1}{16}\{[v^2 - n(n+2)w^2]^2 - 4v^2w^2\}$$

or

(8) $$ac = \frac{1}{16}[v^2 - n^2w^2][v^2 - (n+2)^2w^2]$$

The coefficients of $f'(x)$ may now be determined to give the required rational roots in (1) and (2). $b$ is found from (7) but its sign may be taken either plus or minus at will since it occurs only as $b^2$ in (3), (4), and (5). $ac$, found from (8), may be factored at pleasure into $a$ and $c$.

We note from (8) that with this choice of factors in (6) that $ac$ is negative if $n < \left|\frac{v}{w}\right| < n+2$ and $ac$ is positive if $\left|\frac{v}{w}\right|$ is outside that range.

Some examples follow. $f(x)$ a quartic, $n = 1$. Let $v = 5$, $w = 3$. $b = -1$, $ac = -56$, $f'(x) = 4x^3 - x^2 - 14x = x(4x+7)(x-2)$ $f''(x) = 12x^2 - 2x - 14 = 2(6x-7)(x+1)$, $f(x) = x^4 - \frac{1}{3}x^3 - 7x^2$. When $ac$ is negative, as here, the zero root falls between the other roots.

Again, $n = 1, \; v = 5, \; w = 1, \; b = -11, \; ac = 24.$

$$f'(x) = 4x^3 - 11x^2 + 6x = x(4x - 3)(x - 2)$$

$$f''(x) = 12x^2 - 22x + 6 = 2(3x - 1)(2x - 3)$$

$$f(x) = x^4 - \frac{11}{3}x^3 + 3x^2.$$

The case when $f(x)$ is a quintic is of interest. Then $n = 2$ and the zero root of (1) produces a horizontal tangent together with a point of inflection. It is between the maximum and minimum points and the other points of inflection if $ac$ is negative.

Some examples: $n = 2, \; v = 6, \; w = 2. \; b = 2, \; ac = -35.$

$$f'(x) = 5x^4 + 2x^3 - 7x^2 = x^2(5x + 7)(x - 1)$$

$$f''(x) = 20x^3 + 6x^2 - 14x = 2x(10x - 7)(x + 1)$$

$$f(x) = x^5 + \frac{1}{2}x^4 - \frac{7}{3}x^3$$

Again, $n = 2, \; v = 6, \; w = 1. \; b = -14, \; ac = 40.$

$$f'(x) = 5x^4 - 14x^3 + 8x^2 = x^2(5x - 4)(x - 2)$$

$$f''(x) = 20x^3 - 42x^2 + 16x = 2x(5x - 8)(2x - 1)$$

$$f(x) = x^5 - \frac{7}{2}x^4 + \frac{8}{3}x^3$$

We now show one other of the ways in which the factors in (6) may be equated.

Let $q + b = n(n + 2)p, \quad q - b = p.$

Then $q = \dfrac{(n + 1)^2 p}{2}, \quad b = \dfrac{(n^2 + 2n - 1)p}{2}$

$$ac = \frac{1}{16}[(n^2 + 2n - 1)^2 p^2 - 4p^2] = \frac{1}{16}p^2(n + 1)^2(n + 3)(n - 1)$$

This cannot be used for $f(x)$ a quartic since $n = 1$ makes $ac = 0$ but for the quintic $f(x)$ we have with $n = 2$, $b = -\frac{7}{2}p$, $ac = \frac{45}{16}p^2$.

Let $p = 4$, then $b = -14$, $ac = 45$,

$$f'(x) = 5x^4 - 14x^3 + 9x^2 = x^2(5x - 9)(x - 1)$$

$$f''(x) = 20x^3 - 42x^2 + 18x = 2x(5x - 3)(2x - 3)$$

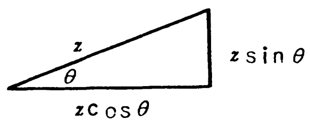
$$f(x) = x^5 - \frac{7}{2}x^4 + 3x^3$$

Of course all these examples can be varied by factoring $ac$ in other ways, or by multiplying through by a constant, or by adding a constant to $f(x)$. There are then indefinitely many solutions to the problem.

Worcester Polytechnic

## PYTHAGOREAN PRINCIPLE AND CALCULUS

Leonard Caners




Consider the rt-angled triangle with θ and $z$ variables.

Let $$z^2\cos^2\theta + z^2\sin^2\theta = u \qquad (1)$$

Then $$\cos^2\theta + \sin^2\theta = \frac{u}{z^2} \text{ or } uz^{-2}$$

Differentiating both sides with respect to $\theta$ we get,

$$-2\cos\theta\sin\theta + 2\sin\theta\cos\theta = D_\theta\,(uz^{-2})$$

$\therefore$ the derivative of $\cos^2\theta + \sin^2\theta = 0$.

Hence, $\cos^2\theta + \sin^2\theta = C$ a constant.

Let $\theta = 0$, then $C = 1$

$$\therefore \cos^2\theta + \sin^2\theta = 1 \qquad (2)$$

Substituting in (2) in (1) $z^2 = u$.

$\therefore z^2 =$ sum of sides of triangle squared, which establishes the Pythagorean theorem.

P.S. We do not believe a petitio principii is involved since none of the pre-requisite proofs involves the Pythagorean theorem. Thus $\lim_{\theta\to 0} \frac{\sin d\theta}{d\theta} = 1$; the binomial theorem etc. can all be established independently.

St. Michael's College

# TEACHING OF MATHEMATICS

*Edited by*

Joseph Seidlin and C. N. Shuster

This department is devoted to the teaching of mathematics. Thus articles on methodology, exposition, curriculum, tests and measurements, and any other topic related to teaching, are invited. Papers on any subject in which you, *as a teacher*, are interested, or questions which you would like others to discuss, should be sent to Joseph Seidlin, Alfred University, Alfred, New York.

## THE GROUP METHOD

S. Birnbaum and K. Ommidvar

The constant stream of articles directed towards the improvement of mathematics teaching attests to the difficulty of the subject. It may also suggest that no really satisfactory solution for the problem of "getting Math. across" has been attained. (By 'satisfactory' we mean that the Math. student, at the end of his first college Math. course would have about the same insight into Math. as the Chem. student has into chemistry via the Periodic Table of Elements.)

Almost everyone is now agreed that the best teaching tries to give the student insight into and understanding of the topics treated. Not everyone is agreed that the way to attain this difficult goal is to reduce the number of topics treated. At a time when mathematics is expanding as never before, it may not seem wise to reduce the number of topics discussed, even on an elementary level.

To the credit of Math. teachers as a whole, it must be admitted that most suggestions for improvement in Math. schoolroom work have been directed towards increasing the responsibility of the Math. teacher.

We would here like to make an additional recommendation aimed at resulting in more students learning Math. better. We do not believe that out recommendation is the "gimmick" which will forever solve the problem. However, the idea to be presented seems to flow right out of the gregarious and social nature of human beings.

Our suggestion for improvement is aimed as much at the student as it is at the teacher. It consists in the idea that we *combine dynamic group activity with individual responsibility*.

Why?

Admittedly, Math. is difficult for most students. This difficulty is reduced by having capable teachers. And if these capable teachers really like the young students so much the better; for then, they avoid the student failure which results from lack of response to an unsympathetic teacher. Yet something more is needed than the combination of the capable, sympathetic teacher and the present-day trend of constant improvement teaching techniques. In spite of these improvements, difficulties still persist.

In response to this, it is urged that part of the answer lies in having recourse to the method by which mankind has been able to resolve a great many difficulties: *group activity*.

How would the combination of group activity and individual responsibility be applied in a classroom situation?

To begin with, instead of regarding the class as composed only of individuals, the teacher would have to think of it as a unit. And this unit would itself be, not a collection of individuals, but a unit of groups. In other words, the teacher would have to divide his class up into four or five groups. (If his students can do this themselves so much the better!) Then each group would choose its own leader. The purpose of the group is twofold: 1) to have the group assist each one of its members in learning Math; 2) by such assistance and by mere membership itself to give the student a feeling of confidence, of belonging, and of assurance that he will be able to get his work done. With the surety which comes from many heads being better than one, the poorer student will overcome his fear of Math. and will do much better than if left on his own.

Even today, without this conscious use of group activity there is a tendency among some of the students towards mutual assistance in school work. Properly organized, this tendency can prove of great aid to the teacher. Since the group itself will look for guidance to its most capable Math. student, the teacher will be able to count on four or five unofficial assistants who spend much more time with the other students than the teacher does.

Let's be even more detailed and see how this works out in a practical situation. The earnest teacher spends several hours preparing a lesson. He or she thinks that the presentation has been so clarified that no one can fail to grasp the idea of the particular lesson. So he or she gives his lesson. Aware of the wise dictum that performance is the best way to learn, the teacher sends most of the class to the blackboards to work problems connected with the subject under discussion. The students go to the board as groups, all the members of the group adjacent to each other. Everyone solves problems; but when poor Joe Slow reaches an impasse, he turns to a member or the members of his group and they help him out. If they can't, the teacher rushes to the rescue and all becomes well again. Besides, the teacher has the satisfaction of knowing that he has acted to remove a possible cause of failure.

The thoughtful reader will have perceived some additional advantages of the method, not yet mentioned, such as bringing the teacher down from his dais to give more personalized instruction. In other words, the group method would result in better contact betweenthe teacher and the student. Oftentimes, an individual who fails to understand may hesitate to ask the teacher for clarification. But he will rarely hesitate to ask the group and the group will not hesitate to ask the teacher. Then think of the economy of effort for the teacher! Even the hardest working teacher cannot possibly give personal attention to each and every student on each and every question in the student's mind. But by having the group itself resolve the simpler questions the teacher is able to act as a reserve free to clear up the more difficult problems which are now of interest to a whole group and perhaps more than just one group.

How about possible disadvantages? Will this method be an encouragement for the shirker? No, we think not. Groups of individuals have a way of developing their own disciplinary mechanisms without outside interferance. But it would certainly help if the teacher defined the duties and rights of the group in advance, and made it clear that the group should expect every man to do as much as he can. It should be understood that the opinion of the group about an individual would be important to the teacher.

How about homework?

We must recognize that a certain amount of collaboration is already the rule in this domain. And indeed, wherever the students are graded for homework the advantage lies with the student who gets help from classmates as against the purely rugged individualist. The group method thus starts everybody off even so far as homework is concerned. But by recognizing and even furthering group application the teacher is able to try to make the work individual even though the preparatory work is of a group nature. Without such recognition, the teacher is limited to 'guerilla warfare' against copying.

Clearly, the intention of the group method is to improve the work of the poorer students. And such intention is precisely in harmony with the democratic viewpoint. On the other hand it does not conflict with the hope of developing the best students. For, by giving such students an opportunity to take a more active part in the classroom work it enables them to show their mettle and bring themselves to the teacher's attention.

The *dynamic group method* has been applied, at least in some graduate courses in Education in a New York University. Why not give it a test on the lower levels?

Brooklyn Polytechnic Institute

# MOTIVATING THE STUDY OF DETERMINANTS

*W. L. Shepherd*

In teaching freshmen to use determinants to solve systems of three linear equations in three unknowns, I find a number of moderately able students who, having rather thoroughly mastered the technique of obtaining two equations in two unknowns, etc., are unwilling to consider other methods. I have had some success in obtaining the interest of a considerable number of such students by using discussions similar to the following.

Let us solve simultaneously the equations

$$5x - 2y = 13,$$
$$4x - 7y = 5.$$

We eliminate $y$ by using 7 and -2 as multipliers, and adding, to obtain

$$35x - 14y = 91,$$
$$-8x + 14y = -10,$$
$$27x = 81,$$
$$x = 3.$$

Likewise we can obtain $y = 1$.

Notice that when we were thinking of eliminating $y$ in order to obtain $x$ we could have been thinking of eliminating all of the unknowns but $x$. Perhaps we can solve systems of three equations in three unknowns, say $x$, $y$ and $z$, by finding three multipliers which will permit us to use addition to eliminate all of the unknowns but $x$. The following example will show that this is sometimes possible.

*Example.* Solve simultaneously the system

$$2x + y - 3z = 11,$$
$$3x + 2y - 3z = 16,$$
$$x + 3y - 2z = 11.$$

Using 5, $-7$ and 3 as multipliers we obtain

$$10x + 5y - 15z = 55,$$
$$-21x - 14y + 21z = -112,$$
$$3x + 9y - 6z = 33.$$

Addition gives

$$-8x + 0 \cdot y + 0 \cdot z = -24,$$
$$x = 3.$$

We conclude that if the system has a solution, the $x$-value of this solution must be 3.

For convenience in later work let us summarize the arithmetic of the above work as

$$(2\cdot5 - 3\cdot7 + 1\cdot3)x + (1\cdot5 - 2\cdot7 + 3\cdot3)y$$
$$+ [(-3)\cdot5 - (-3)\cdot7 + (-2)\cdot3]z = 11\cdot5 - 16\cdot7 + 11\cdot3.$$

We have as yet left unexplained how we obtained the multipliers 5, -7 and 3, or how we may go about finding multipliers which would enable us to eliminate $x$ and $y$, say, in order to obtain $z$. Clearly, these multipliers should be such that, after adding, we obtain the coefficients of both $x$ and $y$ to be zero. In some cases we might be able to discover them by trial and error. After studying what our authors have to say about solving such systems of equations by determinants we shall be able to return to this example and point out the systematic procedure by means of which we may obtain these multipliers.

Texas Western College

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## NOTES ON A FRACTION PROBLEM IN COLLEGE ALGEBRA

Hazel Schoonmaker Wilson

A recent assignment in College Algebra included the following problem taken from Gordon Fuller's *College Algebra,* Third Printing, p. 54:

> " The denominator of a fraction is one more than the numerator. If the numerator is increased by 10 and the denominator by 15, the value of the fraction is unchanged. Find the fraction.''

One of the students gave as his equation for this problem:

$$\frac{x}{x+1} = \frac{10}{15}$$

Solving this equation gives $x = 2$ and the value of the fraction as 2/3 which is correct. When asked for an explanation the only comment the student made was "It works".

The usual equation for this problem is

$$\frac{x}{x+1} = \frac{x+10}{x+1+15}\,.$$

If we consider the general case we have for our equation:

$$\frac{x}{x+a} = \frac{x+b}{x+a+c}$$

Clearing of fractions gives $cx = ab+bx$, or $cx = b(a+x)$. Then

$$(1) \qquad \frac{x}{x+a} = \frac{x+b}{x+a+c} .$$

The student's equation is correct. From (1) it is obvious that the quantities to be added to the numerator and denominator of the required fraction are proportional to the numerator and denominator.

Solving (1) for $x$, gives

$$x = \frac{ab}{c-b} .$$

In problems of this type it is usually assumed that all the quantities $x$, $a$, $b$, and $c$ are positive integers. If this assumption is made then the quantities $a$, $b$, and $c$ cannot all be chosen arbitrarily. It is necessary and sufficient that $c-b$ is a factor of $ab$. It is obvious that $b$ must be chosen less than $c$ and that the required fraction will be proper.

Jacksonville State College, Alabama

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Advance plans for the Fifth Annual Conference for Teachers of Mathematics and a Fourth Annual Mathematics Laboratory on the Los Angeles campus of the University of California have been announced by University Extension, with conference dates sat for July 5 to 15.

University Extension will present the sessions in cooperation with the Departments of Mathematics and Education at U.C.L.A., the California Mathematics Council, and the National Council of Teachers of Mathematics.

The Laboratory will meet daily during morning hours and the Conference during afternoon hours. Both are open to all teachers or prospective teachers of at least Senior standing. Fees set are \$20 for each laboratory, or \$30 for two study groups, or \$40 for one laboratory and one or more study groups. General sessions will be open to any enrollee, irrespective of the fee paid.

Additional information is available on request to University of California Extension, Los Angeles 24.

Clifford Bell

Director of the Conference
University of California,
Los Angeles, Calif.

# PROBLEMS AND QUESTIONS

Edited by

Robert E. Horton, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new and subject matter questions that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted.

Solutions should be submitted on separate, signed sheets. Figures should be drawn in India ink and twice the size desired for reproduction.

Send all communications for this department to *Robert E. Horton, Los Angeles City College, 855 N. Vermont Ave., Los Angeles 29, California.*

## PROPOSALS

**236.** *Proposed by C. W. Trigg, Los Angeles City College.*

What is the largest value of $y$ such that there is a binomial expansion in which the coefficients of $y$ consecutive terms are in the ratio 1:2:3: ... :$y$? Identify the corresponding expansion and the terms.

**237.** *Proposed by M.N.Gopalan, Maharaja's College, Mysore, South India.*

Let $p_1$, $p_2$, $p_3$ be the altitudes and $q_1$, $q_2$, $q_3$ the medians of a triangle $ABC$. Prove:

(1) If $p_1p_2 + p_2p_3 + p_3p_1 = q_1q_2 + q_2q_3 + q_3q_1$ the triangle is equilateral.

(2) If $R(\tan A + \tan B + \tan C) = 2s$ where $R$ is the circumradius and $s$ is the semi-perimeter then $ABC$ is equilateral.

(3) If $p_1 + p_2 + p_3 = 9r$, where $r$ is the inradius, then $ABC$ is equilateral.

**238.** *Proposed by Chih-yi Wang, University of Minnesota.*

Solve the differential equation:
$\frac{dx}{dt} = ax^3 + bx + c$, $x = x_0 \neq 0$ when $t = 0$ where $a$, $b$ and $c$ are real constants. Discuss the values of $a$, $b$ and $c$ which will give a stable solution, that is, in the solution, $x$ remains bounded as $t$ becomes infinite.

**239.** *Proposed by Norman Anning, Alhambra, California.*

In a certain solar system a planet has a moon. The position of the moon with respect to the sun of the system is an epicycle.

$$x = a\cos t + b\cos nt$$
$$y = a\sin t + b\sin nt$$

where $t$ is a parameter.

a) If $a = 93$, $b = .24$ and $n = 13$ show that the path of the moon is everywhere concave to the sun.

b) Suppose the moon gets a shot of new energy. If $a = 93$, $b = .24$ and $n = n$, what is the smallest $n$ for which the path of the moon will have points of inflection?

**240.** *Proposed by M.S.Klamkin, Polytechnic Institute of Brooklyn.*

Determine the value of

$$A\times(B\times C)+B\times(C\times A)+C\times(A\times B)$$

without expanding any of the triple vector products.

**241.** *Proposed by Leon Bankoff, Los Angeles, California*

A thin rod of length $L$ is the longest that can be moved horizontally from one corridor into another at right angles to the first. When the rod touches the inner corner and the two outer walls of the corridors, its inclination to the two walls is $\theta$ and $(90^\circ-\theta)$. Given $L$ and $\theta$ find $a$ and $b$, the widths of the two corridors.

**242.** *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Let $A'$, $B'$, $C'$ be the points dividing the sides of triangle $ABC$ in the ratio $k$, and let $A''$, $B''$, $C''$ be the points dividing the sides of triangle $A'B'C'$ in the ratio $1/k$. Prove that the triangle $A''B''C''$ is homothetic with the original triangle $ABC$.

## SOLUTIONS

### LATE SOLUTIONS

**207.** *C. S. Raman, Mysore, South India.*

**208.** *M.N. Gopalan, Maharaja's College, Mysore, South India; M. S. Klamkin, Polytechnic Institute of Brooklyn.*

**214.** *Huseyin Demir, Zonguldak, Turkey.*

### DIOPHANTINE ARC COTANGENTS

**215.** [November 1954] *Proposed by Norman Anning, Alhambra, California.*

Solve arc cot$x$ + arc cot$y$ = arc cot$z$ where $x$, $y$ and $z$ are integers such that $x > y > z > 0$.

*Solution by Walter B. Carver, Cornell University.*

Since

$$\text{arc cot}\, x + \text{arc cot}\, y = \text{arc cot}\, \frac{xy - 1}{x + y},$$

the given equation is equivalent to:

$$\frac{xy - 1}{x + y} = z \quad \text{or} \quad (x - z)(y - z) = z^2 + 1.$$

Hence we take an arbitrary positive integer for $z$, and express $z^2 + 1$ as the product of two positive integers, $z^2 + 1 = hk$, $h > k$. (For any $z$ one such pair of factors is $h = z^2 + 1$, $k = 1$). Then a solution is:

$$x = z + h, \quad y = z + k, \quad z,$$

and all solutions can be obtained in this way. For each value of $z$ there is at least one solution, and as many solutions as there are ways of expressing $z^2 + 1$ as a product of two factors.

For example, with $z = 17$ there are three solutions; but with $z = 20$, $z^2 + 1$ is a prime and there is only one solution.

*Also solved by Leon Bankoff, Los Angeles, California; H. M. Feldman, St. Louis, Missouri; John Jones Jr., Mississippi Southern College; C. E. Jones and L. J. Venable (jointly), Tennessee A and I State University, Nashville, Tennessee; M. S. Klamkin, Polytechnic Institute of Brooklyn; Louis S. Mann, Gardena, California; George Mott, Repbulic Aviation Corporation, New York; L. A. Ringenberg, Eastern Illinois State College; S. H. Sesskin, Hofstra College, New York; Harry Siller Far Rockaway, New York; A. Sisk, Maryville College, Tennessee; E. P. Starke, Rutgers University; Chih-yi Wang, University of Minnesota; Hazel S. Wilson, Jacksonville State College, Alabama;* and the proposer.

Sesskin pointed out that the Fibonacci series 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 leads to some of the solutions (1, 2, 3); (3, 5, 8); (8, 13, 21); (21, 34, 55); (55, 89, 144).

Jones and Venable tabulated all the solutions through $z = 9$

| | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | 2 | 3 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 7 | 8 | 8 | 9 | 9 |
| $y$ | 2 | 3 | 4 | 5 | 5 | 6 | 7 | 7 | 8 | 9 | 12 | 9 | 13 | 11 | 10 |
| $x$ | 3 | 7 | 13 | 8 | 21 | 31 | 18 | 43 | 57 | 32 | 17 | 73 | 21 | 50 | 91 |

## A FRACTIONAL SUM

**216.** [November 1954] *Proposed by Erich Michalup, Caracas, Venezuela.*

Prove that

$$\sum_{n=1}^{\infty} \frac{16n^2 + 12n - 1}{8(4n + 3)(4n + 1)(2n + 1)(n + 1)} = \frac{1}{24}$$

I. *Solution by Dennis C. Russell, Birkbeck College, University of London.*

If the general term of the series be put into partial fractions, we have:

$$a_n \equiv \frac{16n^2+12n-1}{8(4n+3)(4n+1)(2n+1)(n+1)} = \frac{-1/2}{4n+1} + \frac{3/2}{4n+2} - \frac{1/2}{4n+3} - \frac{1/2}{4n+4}.$$

Since $\sum a_n x^n$ has a radius of convergence 1, it follows by Abel's theorem on power series that

$$\sum_{n=0}^{\infty} a_n = \lim_{x\to 1-} \sum_{n=0}^{\infty} a_n x^n$$

Now

$$\sum_{n=0}^{\infty} a_n x^{4n+4} = -\frac{x^3}{2}\sum_{n=0}^{\infty}\frac{x^{4n+1}}{4n+1} + \frac{3x^2}{2}\sum_{n=0}^{\infty}\frac{x^{4n+2}}{4n+2}$$

$$-\frac{x}{2}\sum_{n=0}^{\infty}\frac{x^{4n+3}}{4n+3} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^{4n+4}}{4n+4}$$

$$= -\frac{x^3}{2}\int_0^x \frac{dt}{1-t^4} + \frac{3x^2}{2}\int_0^x \frac{t}{1-t^4}\,dt - \frac{x}{2}\int_0^x \frac{t^2}{1-t^4}\,dt$$

$$-\frac{1}{2}\int_0^x \frac{t^3}{1-t^4}\,dt - \frac{1}{2}\int_0^x \frac{(x-t)(x^2-2t-t^2)}{(1-t)(1+t)(1-t^2)}\,dt.$$

As $x$ approaches 1- this integral approaches

$$-\frac{1}{2}\int_0^1 \frac{1-2t-t^2}{(1+t)(1+t^2)}\,dt \quad -\frac{1}{2}\int_0^1 \frac{1}{1+t} - \frac{2t}{1+t^2}\,dt$$

$$= -\tfrac{1}{2}\left[\log(1+t) - \log(1+t^2)\right]_0^1 = 0$$

Hence

$$\sum_{n=0}^{\infty} a_n = 0 \text{ so that } \sum_{n=1}^{\infty} a_n = -a_1 = -(-1/24) = 1/24.$$

II. *Solution by L.A.Ringenberg, Eastern Illinois State College.*

Express each term of the series $S$ as a sum of four fractions with

denominators $4n + 1$, $4n + 2$, $4n + 3$, and $4n + 4$.
Then

$$S = \frac{1}{2}\left(-\frac{1}{5}+\frac{3}{6}-\frac{1}{7}-\frac{1}{8}-\frac{1}{9}+\frac{3}{10}-\frac{1}{11}-\frac{1}{12}-\cdots\right).$$

Let

$$S^* = -1 + \frac{3}{2}-\frac{1}{3}-\frac{1}{4}-\cdots+\left(\frac{-1}{4n+1}+\frac{3}{4n+2}-\frac{1}{4n+3}-\frac{1}{4n+4}\right)+\cdots .$$

Then

$$S^* = \left(-1+\frac{3}{2}-\frac{1}{2}-\frac{1}{8}-\cdots\right)+\left(-\frac{1}{3}+\frac{3}{6}-\frac{1}{12}-\frac{1}{24}-\cdots\right)+\cdots$$
$$+\frac{-2}{4n+2}+\frac{3}{4n+2}-\frac{1}{8n+4}-\frac{1}{16n+8}-\cdots+\cdots .$$

But $S^* = 0 - 0 - \cdots - 0 - \cdots = 0$, and since $2S = S^* + 1 - \frac{3}{2} + \frac{1}{3} + \frac{1}{4}$, it follows that $2S = 0 + \frac{1}{12}$, $S = \frac{1}{24}$.

III. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* Let $S$ represent the sum

$$\frac{1}{2}\sum_{1}^{\infty}\left[\frac{3}{4n+2}-\frac{1}{4n+1}-\frac{1}{4n+3}-\frac{1}{4n+4}\right]$$

Now

$$\sum_{1}^{N}\frac{1}{a+nb}=\frac{1}{b}\left[\chi\left(\frac{a}{b}+n+1\right)-\chi\left(\frac{a}{b}+1\right)\right]$$

where $\chi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$. Thus $S = \dfrac{1}{8}\left[-3\chi\left(\frac{3}{2}\right)+\chi\left(\frac{5}{4}\right)+\chi\left(\frac{7}{4}\right)+\chi(2)\right]$

Since $\chi(x+1) - \chi(x) = 1/x$ and $\chi(1) = -\gamma$, $\chi(\tfrac12) = -\gamma - 2\log 2$, $\chi(\tfrac14) = -\pi/2 - \gamma - 3\log 2$, $\chi(\tfrac34) = \pi/2 - \gamma - 3\log 2$ it follows that $S = \frac{1}{24}$

*Also solved by H. M. Feldman, St. Louis, Missouri; John M. Howell, Los Angeles City College; E. P. Starke, Rutgers University; Chih-yi Wang, University of Minnesota* and the proposer.

## PTOLEMY'S THEOREM

**217.** [November 1954] *Proposed by Huseyin Demir, Zonguldak, Turkey.*

Prove that a necessary and sufficient condition for the convex polygon $A_1 A_2 A_3 A_4$ to be inscriptable is that:

$$D = \begin{vmatrix} A_1A_1 & A_1A_2 & A_1A_3 & A_1A_4 \\ A_2A_1 & A_2A_2 & A_2A_3 & A_2A_4 \\ A_3A_1 & A_3A_2 & A_3A_3 & A_3A_4 \\ A_4A_1 & A_4A_2 & A_4A_3 & A_4A_4 \end{vmatrix} = 0$$

where $A_{ij}$ denotes the distance between the vertices $A_i$ and $A_j$ if $j > i$, and $A_jA_i = -A_iA_j$.

*Solution by H. M. Feldman, St. Louis, Missouri*

Since $A_iA_i$ must clearly be zero, the determinant is skew-symmetric and its value is

$$[(A_1A_2)(A_3A_4) + (A_1A_4)(A_2A_3) + (A_1A_3)(A_2A_4)]^2$$

The vanishing of the expression within the brackets is a necessary and sufficient condition for the quadrilateral to be inscriptable in a circle (Ptolemy's Theorem).

*Also solved by Ben K. Gold, Los Angeles City College; M. S. Klamklin, Polytechnic Institute of Brooklyn; E. P. Starke; Rutgers University; Chih-yi Wang, University of Minnesota* and the proposer.

## A PRODUCT OF TWO BINOMIALS

**218.** [November 1954] *Proposed by Ben K. Gold, Los Angeles City College.*

Prove:

$$\sum_{i=0}^{K} (-1)^i \binom{K+i}{K}\binom{2K+1}{K-i} = 1$$

I. *Solution by Harry Siller, Far Rockaway, New York.*

If we expand

$$(1+x)^{-(k+1)} = 1-(k+1)x+\cdots+\frac{(-1)^i(k+1)(k+2)\cdots(k+i)}{i!}x^i+\cdots$$

and

$$(1+x)^{(2k+1)} = 1+(2k+1)x+\cdots+\frac{(2k+1)(2k)\cdots(k-i+2)}{(k-i)!}x^{k-1}+\cdots$$

we see that the required sum

$$\sum_{i=0}^{k} (-1)^i \binom{k+i}{k}\binom{2k+1}{k-i}$$

is equal to the coefficient of $x^k$ in the expansion of $(1+x)^{-(k+1)}(1+x)^{(2k+1)}$ or $(1+x)^k$. But this coefficient is 1 therefore the proposition holds.

II. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.*

We can establish the equality

$$\sum_{r=0}^{k} (-1)^r \binom{k+r}{k}\binom{2k+1}{k-r} = \frac{(2k+1)!}{(k!)^2} \sum_{r=0}^{k} (-1)^r \frac{\binom{k}{r}}{k+1+r}$$

A more general expansion follows immediately from the expansion of the Beta Function (See Authors note in *Scripta Mathematica*, December 1953, p 275). If $m$ and $n$ are non negative integers

$$B(m+1, n+1) = \frac{m!\ n!}{(m+n+1)!} = \int_0^1 t^m(1-t)^n dt = \int_0^1 \sum_{r=0}^{n} (-1)^r \binom{n}{r} t^{m+r}\, dt$$

Thus

$$\sum_{r=0}^{n} \frac{(-1)^r \binom{n}{r}}{m+r+1} = \sum_{r=0}^{m} \frac{(-1)^r \binom{m}{r}}{n+r+1} = \frac{m!\ n!}{(m+n+1)!}$$

The proposed identity follows by setting $m = n = k$. (This identity was previously obtained by R. Gloden, *Scripta Mathematica*, 1952, p 178).

The identity can be extended to non integers $m$ and $n$ both $\geq 0$. In this case the limits are from 0 to $\infty$, and

$$\frac{m!\ n!}{(m+n+1)!} \text{ becomes } \frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}.$$

*Also solved by Huseyin Demir, Zonguldak, Turkey; J. M. Howell, Los Angeles City College; Dennis C. Russell, Birkbeck College, University of London; E. P. Starke, Rutgers University, Chih-yi Wang, University of Minnesota* and the proposer.

## AN HYPERBOLA THROUGH MID POINTS

**219.** [November 1954] *Proposed by N. Shklov, University of Saskatchewan.*

Let a straight line through the origin meet the lines $x + y - 4 = 0$ and $x - y - 4 = 0$ in the points $A$ and $B$ respectively. Let $M$ be the mid point of the segment $AB$. Determine the locus of $M$ as the line $0AB$ is rotated about 0.

*Solution by W. M. Sanders, Mississippi Southern College.*

Represent line $0AB$ by $y = mx$ and the point $M$ by $(x, y)$. It is observed that $A{:}(4/(1+m),\ 4m/(1+m))$; $B{:}(4/(1-m),\ 4m/(1-m))$; and $C{:}(4,0)$ are vertices of a right triangle having $M$ as its circumcenter. Hence $AM = CM$. It follows that

$$(x - \frac{4}{1+m})^2 + (y - \frac{4m}{1+m})^2 = (x-4)^2 + y^2$$

where $m = y/x$. Simplification yields the hyperbola

$$x^2 - 4x - y^2 = 0$$

as the locus.

*Also solved by Walter B. Carver, Cornell University; Huseyin Demir, Zonguldak, Turkey; Abraham L. Epstein, Cambridge Research Center, Massachussetts; Clinton E. Jones, Tennessee A and I State University; M. A. Kirchberg, Hopkins, Michigan; M. S. Klamkin, Polytechnic Institute of Brooklyn; Louis S. Mann, Gardena, California; M. Morduchow, Polytechnic Institute of Brooklyn; W. Moser, University of Toronto; George Mott, Republic Aviation Corporation; T. F. Mulcrone, St. Charles College, Louisiana; Lawrence A. Ringenberg, Eastern Illinois State College; Dennis C. Russell, Birkbeck College, University of London; S. H. Sesskin, Hofstra College, New York; Harry Siller, Far Rockaway, New York; A. Sisk, Maryville College, Tennessee; E.P. Starke, Rutgers University; Sister M. Stephanie, Georgian Court College, Lakewood, New Jersey; Chih-yi Wang, University of Minnesota; Hazel S. Wilson, Jacksonville State College, Alabama* and the proposer.

## THE ODDNESS OF FAREY SEQUENCES

**220.** [November 1954] *Proposed by Thomas F. Mulcrone, St. Charles College, Louisana.*

Show that for $n \geq 2$ the Farey sequence $F_n$ contains an odd number of terms.

I. *Solution by E. P. Starke, Rutgers University.*

The sequence $F_n$ consists of all rational fractions $a/b$ with $a$, $b$ relatively prime and $0 < a < b < n$. If $a/b$ is one term, then $(b-a)/b$ is another except when $a = b - a$, i.e., when $a = 1$, $b = 2$. Thus $F_n$ contains the term ½ and other terms all of which can be paired—hence an odd number of terms.

II. *Solution by Chih-yi Wang, University of Minnesota.*

For $n = 2$, $F_2 : 0/1, 1/2, 1$ and for $n = 3$, $F_3 : 0/1, 1/3, 1/2, 2/3, 1$, and assume that the statement is true for the sequence $F_n$. We want to prove that it is also true for the sequence $F_{n+1}$. The number of terms of the sequence $F_{n+1}$ differs from that of $F_n$ by the total number of $k$'s, such that $(k, n+1) = 1$ and $0 < k < n+1$, which is precisely Euler's $\phi$ function or totient of $n+1$, in symbols, $\phi(n+1)$. The fact that $\phi(n+1)$ is even for $n \geq 2$ follows from the two well known theorems:

Theorem 1: If $p$ is a prime, $\phi(p^e) = p^e - p^{e-1}$.

Theorem 2: If $a$ and $b$ are relatively prime positive integers $\phi(ab) = \phi(a)\phi(b)$.

Hence the statement is true by induction.

*Also solved by Walter B. Carver, Cornell University; Huseyin Demir, Zonguldak, Turkey; M. S. Klamkin, Polytechnic Institute of Brooklyn* and the proposer.

## SPIRAL ARC LENGTH

**221.** [November 1954] *Proposed by E.P.Starke, Rutgers University.*

On a conical surface there is traced a spiral which crosses each of the linear elements at a fixed angle $\psi$. Find a simple expression for the length of this spiral between any two of its points.

*Solution by Huseyin Demir, Zonguldak, Turkey.*

The cone is a developable surface. When developed the $\psi$-spiral on the cone is transformed into a $\psi$-logarithmic spiral on the plane, of which the polar equation is:

$$r = a e^{(\cot \psi)\theta}$$

Then
$$ds = \sqrt{dr^2 + r^2 d\theta^2} = \frac{a}{\sin \psi} e^{(\cot \psi)\theta} d\theta.$$

Between two points on the spiral

$$s = \frac{a}{\sin\psi}\int_a^b e^{(\cot \psi)\theta} d\theta = \frac{a}{\cos \psi}\left| e^{(\cot \psi)\theta}\right|_a^b = \frac{|r|_a^b}{\cos \psi}$$

$s = (b - a)/\cos \psi$ where $a$ and $b$ denote the distances of the points from the vertex of the cone.

*Also solved by Walter B. Carver, Cornell University; M. S. Klamkin, Polytechnic Institute of Brooklyn; S. H. Sesskin, Hofstra College, New York; A. Sisk, Maryville College, Tennessee and* the proposer.

## QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

**Q 140.** If $a^{-1} + b^{-1} \pm c^{-1} = (a + b + c)^{-1}$ show that $a^{-n} + b^{-n} + c^{-n} = (a^n + b^n \pm c^n)^{-1}$. [*Submitted by Barney Bissinger.*]

**Q 141.** If the twelve months of the year are written in the order offered by $5n + 2 \pmod{12}$, $n = 1, 2, 3, \ldots 12$, what can be said about the characteristics of the first seven, the next four, and of the last month? [*Submitted by Huseyin Demir.*]

**Q 142.** Find the class of functions such that $\frac{1}{F(x)} = F(-x)$. One simple example is $F(x) = e^x$. [*Submitted by M. S. Klamkin.*]

**Q 143.** Show that ½ is the value of

$$\int_0^1 \int_{\sqrt{y}}^1 e^{y/x}\, dx\, dy$$

[*Submitted by E. P. Starke.*]

**Q 144.** Sum $\tan\theta + \frac{1}{2}\tan\theta/2 + \ldots + [1/(2^n) \tan\theta/2^n]$
[*Submitted by M. S. Klamkin*]

**Q 145.** Determine the equation of the cone through the origin passing through the intersection of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.
[*Submitted by M.S.Klamkin.*]

**ANSWERS**

**A 140.** Clearing fractions one sees that $a = -b$, $b = \pm c$ or $a = \pm c$, each of which makes the corresponding result an identity.

**A 141.** The first seven months listed will have 31 days, the next four months will have 30 days and the last one has 29 or 28 days.

**A 142.** Let $F(x) = E(x) + O(x)$ where $E(x)$ is even and $O(x)$ is odd.

Then $$\frac{1}{E(x) + O(x)} = E(x) - O(x)$$

Thus $E(x) = \pm\sqrt{1 - O(x)^2}$ and $F(x) = O(x) \pm \sqrt{1 - O(x)^2}$

Let $O(x) = \tan x$, then $F(x) = \tan x \pm \sec x$.

**A 143.** By interchanging the order of integration we have:

$$\int_0^1 \int_{\sqrt{y}}^1 e^{y/x}\, dx\, dy = \int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx$$

which is easily integrated and evaluated to be ½.

**A 144.** Since $\tan\theta = \cot\theta - 2\cot 2\theta$ the sum equals

$$\lim_{n\to\infty}\left[\frac{1}{2^{n-1}}\cot\frac{\theta}{2^{n-1}} - 2\cot 2\theta\right] = 1/\theta - 2\cot 2\theta$$

**A 145.** The surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \left[\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right]^2$ is a cone through the origin and obviously passes through the given intersection.

## TRICKIES

A trickie is a problem whose solution depends upon the perception of the key word, phrase or idea rather than upon a mathematical routine. Send us your favorite trickies.

**T 16.** Determine $\theta$ such that

$$\frac{\sin\theta + \sin 2\theta + \sin 3\theta}{\cos\theta + \cos 2\theta + \cos 3\theta} = \tan 2\theta$$

[*Submitted by M. S. Klamkin.*]

**T 17.** Find the relationship between $A$ and $B$ if

$$A = 1 + \frac{2}{1!} - \frac{2^3}{3!} + \frac{2^5}{5!} - \frac{2^7}{7!} + \cdots \text{ and } B = 2 - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + \frac{10}{9!} - \cdots$$

[*Submitted by M. S. Klamkin*]

**T 18.** Evaluate $\cos 5^\circ + \cos 77^\circ + \cos 149^\circ + \cos 221^\circ + \cos 293^\circ$.

[*Submitted by M. S. Klamkin.*]

## SOLUTIONS

**S 16.** As this is an identity it is satisfied for all $\theta$ for which the denominator is not zero.

**S 17.** $A = 1 + \sin 2$ and $B = \sin 1 + \cos 1$. Therefore, $A = B^2$

**S 18.** The expression is the sum of the projections of a regular pentagon and therefore equals zero.

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Dear Mr. James,

I was very pleased to find in the December issue of your MAGAZINE a solution to the Food Index Function for which I asked one year ago (question no. 193) and should like to congratulate Mr. Herrera of Los Angeles City College on his excellent work.

Sincerely yours,

Francis Joseph Weiss

# CURRENT PAPERS AND BOOKS

*Edited by*

H. V. Craig

This department will present comments on papers previously published in the MATHEMATICS MAGAZINE, lists of new books, and book reviews.

In order that errors may be corrected, results extended, and interesting aspects further illuminated, comments on published papers in all departments are invited.

Communications intended for this department should be sent in duplicate to H. V. Craig, Department of Applied Mathematics, University of Texas, Austin 12, Texas.

*The Elements of Probability Theory and Some of Its Applications.* By Harald Cramér, John Wiley & Sons, 440 Fourth Ave., New York 16, New York, 1955, 281 pages, $7.00. This is the lastest addition to the Wiley Publications in Statistics, edited by Walter A. Shewhart and S.S.Wilks.

After providing an historical introduction, Dr. Cramér has arranged his material in three major divisions. The first covers the foundations, with definitions, rules, and some basic applications. Part II is devoted to random variables and probability distributions. This section includes discussions of variables and distributions in one dimension, the binomial and related distributions, the normal distribution, further continuous distributions, and variables and distributions in two and more dimensions. In Part III, applications are treated exclusively, making this section of particular value to biologists, engineers, and all those engaged in the application of statistical methods.

Richard Cook

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*Advanced Mathematics for Engineers.* By. Harry W. Reddick and Frederic H. Miller, John Wiley & Sons, N.Y., February 1955, 548 pp. $6.50.

The indispensable mathematical tools of civil, mechanical, electrical, and chemical engineering have been sharpened again in the new third edition of this book. Dr. Miller, who handled this revision, has retained the major features that made the book a standard, but has worked in a variety of changes that increase its usefulness. An introductory account of Legendre's equation now appears in the chapter entitled *Gamma, Bessel, and Legendre Functions*. The physical applications of Legendre polynomials have been added here as well as in the following chapter on partial derivatives and partial differential equations, where Laplace's equation in spherical coordinates is considered.

Richard Cook

# ROUND TABLE ON FERMAT'S LAST THEOREM

## ON FERMAT'S LAST THEOREM

D. E. Stone

The following theorem, pertaining to solutions of the Diophantine equation $a^n + b^n + c^n = 0$, is proved in this note:

THEOREM: A necessary condition that there exist pairwise relatively prime integers $a$, $b$, and $c$ which satisfy the equation $a^n + b^n + c^n = 0$, when one of the triplet $(a, b, c)$ is divisible by $n$ and $n$ is a prime $> 2$, is that the equation $a^n + b^n + c^n = 0$ be expressible in the form

$$T^n + (Bn^{nq-1} - T)^n + (An^q - Bn^{nq-1})^n = 0$$

where $T$, $A$ and $B$ are integers which are not divisible by $n$, and $q \geq 1$ is an integer.

PROOF: (All letters represent integers unless otherwise indicated)

Applying Fermat's theorem $t^n \equiv t \pmod{n}$ if $n$ is a positive prime to

$$a^n + b^n + c^n = 0, \tag{1}$$

we obtain

$$a + b + c \equiv 0 \pmod{n}. \tag{2}$$

Since it can be shown that $a + b + c \neq 0$,* we may write equation (2) in more general form

$$a + b + c = An^q \tag{3}$$

where $A \not\equiv 0 \pmod{n}$ and $q \geq 1$. By assumption, one of $a$, $b$, $c$ is divisible by $n$, say $a$, so that from equation (2) we must have:

$$b + c = Bn^p \tag{4}$$

where $B \not\equiv 0 \pmod{n}$ and $p \geq 1$.

Using (3) and (4), equation (1) becomes

$$b^n + (Bn^p - b)^n + (An^q - Bn^p)^n = 0 \tag{5}$$

Since $n$ is odd, equation (5) may be written as follows:

$$b^n + (Bn^p - b)^n = (Bn^p - An^q)^n \tag{6}$$

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*If $a + b + c = 0$, then $(a + b)^n = -c^n$. But $a^n + b^n = -c^n$, hence $(a + b)^n = a^n + b^n$ which is impossible for $n \geq 2$.

Expanding the left-hand side of equation (6), we obtain

$$\sum_{k=0}^{n-2} \binom{n}{k}[Bn^p]^{n-k}(-b)^k + b^{n-1}Bn \cdot n^p = [Bn^p - An^q]^n \tag{7}$$

By assumption $b \not\equiv 0 \pmod n$, $B \not\equiv 0 \pmod n$, and $n$ is a positive prime $> 2$, thus $Bb^{n-1} \not\equiv 0 \pmod n$. But $p \geq 1$, therefore

$$\sum_{k=0}^{n-2} \binom{n}{k}(Bn^p)^{n-k}(-b)^k$$

is at least divisible by $n^{2p}$ which is $\geq n^{p+1}$; hence, the left-hand side of equation (7) is exactly divisible by $n^{p+1}$. Applying this to equation (7), we must have

$$\frac{[Bn^p - An^q]^n}{n^{p+1}} = C, \tag{8}$$

where $C \not\equiv 0 \pmod n$.

If $p < q$, then $C$ is given by:

$$\frac{n^{np}}{n^{p+1}}[B - An^{q-p}]^n = C. \tag{9}$$

However, $C \not\equiv 0 \pmod n$, hence $n^{np} \leq n^{p+1}$. But this is a contradiction, since by assumption $p \geq 1$ and $n \geq 3$.

If $p = q$, then $C$ is given by:

$$\frac{n^{nq}}{n^{q+1}}[B - A]^n = C, \tag{10}$$

which is impossible as in the case for which $p < q$.

If $p > q$, then $C$ is given by:

$$\frac{n^{nq}}{n^{p+1}}[Bn^{p-q} - A]^n = C. \tag{11}$$

Since $A \not\equiv 0 \pmod n$ and $C \not\equiv 0 \pmod n$, we must have

$$nq = p + 1. \tag{12}$$

Substituting (12) in equation (5) we obtain

$$b^n + (Bn^{nq-1} - b)^n + (An^q - Bn^{nq-1})^n = 0 \tag{13}$$

which was to be proved.

Naval Research Laboratory, Washington D.C.

# *PROOF OF F.L.T. FOR ALL EVEN POWERS

H. W. Becker

*Dedicated to Miss Annie Davidson, who taught me algebra and geometry at Gresham (Neb.) High School, 1920-21.*

THEOREM: The hypotenuse and odd leg of a Pythagorean triangle (P△) cannot both be $m$th powers, $m > 1$, if the even leg is an $m$th.

Let the odd and even legs and hypotenuse of a primitive P△ be $X$, $Y$, $Z$, with $(P)$ $X^2 + Y^2 = Z^2$ in integers, $(Y)$ $Y = y^m$.

To prove that further $(Z) Z = z^m$, $(X) X = x^m$ cannot hold simultaneously.

PROOF: $(Z)$ requires $z = S^2 + T^2$, since $Z = \boxed{2}$, and if a power, must be that power of a $\boxed{2}$. See [1] p.24-5, 11, [2] p. 168 Volpicelli[27], p.226-7 Vieta[4] and Fermat[10], [3] p.13-4. Then $X = (S - T)_e^{2m}$, $Y = (S - T)_o^{2m}$; subscripts $e, o$ denoting respectively the even and odd powered terms of the binomial expansion. Or $X$, $Y = [(S + jT)^{2m} \pm (S - jT)^{2m}]/2j$, $j = \sqrt{-1}$.

$(X)$ requires $x = U^2 - V^2$, since any odd number is the difference of 2 squares. See [2] p.402-4, and p.738 Euler-Drach[39]. Then $Z = (S + T)_e^{2m}$, $Y = (S + T)_o^{2m}$. Or $Z$, $Y = [(S + T)^{2m} \pm (S - T)^{2m}]/2$.

The above identities are the unique solutions of $(P, Z)$ and $(P, X)$, as is easily proven, the fact being taken for granted in [2] chapters VI and XX.

Assuming $(Z)$ and $(X)$ both hold, $(B)$ $U = S$ and $V = T$ cannot hold simultaneously. Because if they did, then $Y = (S - T)_o^{2m} = (S + T)_o^{2m}$, which is impossible, $T \neq 0$, an equation between 2 polynomials differing only by the alternating minus signs of the former.

Still assuming $Z = z^m$ and $X = x^m$, then $(V)$ there can exist no P△ of hyp. and odd leg $z$ and $x$. Because such $z = S^2 + T^2 = U^2 + V^2$, $x = U^2 - V^2 = S^2 - T^2$. But otherwise than thru $U = S$, $V = T$ which by $(B)$ is inadmissible, $S^2 \pm T^2 = U^2 \pm V^2$ cannot hold for both signs, obviously, or by the identity for numbers equal to sum or else difference of 2 squares in 2 ways, [2] p.355 Euler[66] (11), $e = \mp 1$.

* Presented to the Nebraska Academy of Science and Neb.-S.D. section of the Math. Assn. Of America, Lincoln, Neb., April 23, 1955, condensation of a previous 20-page proof.

But $(V)$ is in direct contradiction to Carmichael's Prob. 9 [1] p.103 $(C_I)$ "when $p$ is an odd prime the equation $x^{2p} + y^{2p} = z^{2p}$, with the condition that $x, y, z$ are prime to $p$, implies the 3 Pythagorean equations $x^2 + y^2 = z_1^2$, $x_1^2 + y^2 = z^2$, $x^2 + y_1^2 = z^2$," and the case II corollary $(C_{II})$. If however $y$ is a multiple of $p^k$, then the first 2 equations are unaltered, but the last equation is replaced by $z^2 - x^2 = p^{2kp-1} y_1^2 = P\square$; and correspondingly when $x$ or $z$ is a multiple of $p^k$. (His $x_1, y_1, z_1$ are all $p$ th powers.) $(C_I)$ and $(C_{II})$ are proved by substituting even exponents into the proof of 'Barlow's Theorem', [1] p.87-9, [2] p.733-4, [3] p.27-8, [4] p.215.

Putting first $m = p$, then any multiple, this contradiction proves the even power F.L.T. for all $m > 2$ not a power of 2. But F.L.T. was proved by Fermat himself for $m = 2$ hence all even $m$; [1] p.18, [2] p.615-6, [3] p.5-6.

$\therefore$ F.L.T. for $2m$th powers is true for all $m \geq 2$. Q.E.D.

COROLLARY: In case I, $XYZ$ prime to $m$, no more than one of $X, Y, Z$ can be an $m$ th power (proof to be completed in a later issue).

*ACKNOWLEDGEMENTS: Thanks are due to the librarians and Math. Depts. of Creighton, Omaha, and Nebraska Universities, and Midland College, for assistance in compiling a 50-item bibliography for the expanded version. And to D. H. Lehmer, who presented me with his father's copy of [1], without which this note would never have been.*

REFERENCES

1. R.D.Carmichael, *Diophantine Analysis* (1915)
2. L.E.Dickson, *History of the Theory of Numbers* (1920) Vol.II
3. L.J.Mordell, *Three Lectures on F.L.T.* (1921)
4. Glenn James, *Mathematics Magazine*, 27 (1954) p.213-6.

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Many years ago I thought I had a proof of Fermat's Last Theorem for all powers, had it typed and arranged with a colleague to read it. Feeling that it was too much to think little me had found a proof for which so many great mathematicians had long sought, I resolved to check all results to which I had made reference. Alas! a certain expansion which I had copied did not procede through the next term as it appeared to do. Thus my paper was buried along with the thousands of similar hopefulls.

I know of no other undertaking in mathematics as tricky as proving Fermat's Last Theorem. So I hope some interested readers will skeptically check the above papers, which incidently, have not been refereed.

Editor

# SEMI-POPULAR AND POPULAR PAGES

## MATHEMATICS AND AUTOBIOGRAPHY

Oliver E. Glenn

Doubtless a mathematician who has reached some seniority in research might well publish some notes on the system of his scientific endeavors, although few have done so. Not only could autobiographical writing do justice historically, perhaps best, to the ideals of a thinker; the clearest estimate of what the future holds for progress ought to follow from a critique of the thought which has animated the individual mathematician*. No harm will be done if, after the lapse of a little time, perhaps, some of his principal results are renewed autobiographically. Also mathematics is humane, presenting the most systematic and effective method of formulation of our theory of reality. A brief autobiography follows.

The lower silurian beach was, literally, the present writer's cradle. It was in the scotch settlement of southern Indiana, a part of the Cincinnati geological uplift where all surface rocks bear many fossils, crinoids, trilobites, brachiopods and many others of the silurian. Moreover in pre-historical times the Illinoian glacier had levelled this region, and then, by melting, had cut the plain into strips, each bounded by rolling hills, the entirety being, in time, densely wooded, and into ravines through which ran crystal streams ever seeking the river Ohio. One growing up in this environment, who did not become a scientist in some degree, evidently failed to heed the plain indications of nature.

This writer's scientific thinking has proceeded in waves, the first of which dealt with the problem of abstract group construction. In 1905 I finished the enumeration of the groups of order $p^2qr$ and had a thesis accepted (Ph.D.), at the University of Pennsylvania in Philadelphia. Subsequent considerations on this field of work revealed some rather formidable obstacles in the way of its further development, although a few important papers on the subject have since appeared. The analysis used for a listing of group-types was a numerical interpretation of an algebra of exponents such as appeared when one transformed a general operator of a self-conjugate sub-group. These exponents rapidly become too complicated to be managable when the order of the group is generalized from simple to complex cases. It appeared to me, and does yet, that abstract group theory might well be under-written, empirically, by a theory of periodicity in lower

* Very true. Sometime back we invited articles on *What Mathematics Means to Me*. Response was very limited. Such sketches as Prof. Glenn's would accomplish about the same end. Editor.

number theory, and in 1928 I published one article of this type (Volume 29, Annals of Mathematics). But for the problem itself, of the construction of abstract groups, one wonders whether all of the relevant generalities are yet known, such as Sylow's theorem. Does Hölder's article on methods (Math. Annalen Bd. 43), reach all of its legitimate objectives?

One mathematical paper leads to another, though the chain may not be endless. Following my paper last mentioned, which was entitled "A generalization of the algebra of the theory of numbers, etc.", I wrote a paper on an extension of the number system to satisfy a theory of congruences for a composite modulus. The Galois Field was generalized, though it did not generalize to a field. It generalized to a complex domain in which, in particular, there may be no primitive roots. The periodicity of powers is characterized by a pair of exponents instead of a single exponent. Every existent congruence has solutions in such a domain. Until very recent times one often saw the assertion that some existent congruences have no solution. I read this paper in Bologna in 1928, before the International Congress of Mathematicians and it was published in their Proceedings.

In the meantime the present writer had generated another wave, on a theory of equations in more than one variable, or the theory of factorable quantics. This subject was, and in fact still is, comparatively new. Work upon it had been published by Cayley, Brill, Gordan, Hadamard, and a good many others. A mathematician won a prize for a paper that dealt with a ternary form of a single numerical order. However, no author had yet identified the simplest relevant algorithm, which proved to be the analytic algorithm of partial fractions, related to the fact that, ordinarily, when a curve degenerates, it is to its asymptotic curves, lines, and points. Following out this principle I found the explicit conditions for linear factorability of the ternary quantic of arbitrary order and a general method for the factorization. This result let me to four other papers in this field, and much remains to be done on factorable quantics. I showed in a recent article that a full generalization of the concept of two surfaces in contact, in $n$-space, will require the use of degenerate or singular quantics as well as of degenerate connections.

It was in 1912 that I made a pilgrimage to the University of Chicago primarily to consult with Professor L. E. Dickson concerning the theory, extensively investigated by him, on the invariants of forms when the transformation is that of the total linear modular group. I found that Dickson was pleased to have such an interview; on the occasion of a meeting of the American Mathematical Society, and at a later time, at his home on Lexington Avenue. He generously suggested that I might well engage in research on the modular problem where, by hypothesis, the quantic would be general and algebraic. To this phase of the modular theory he had made preliminary contributions. From then until about 1950 I, and a number of Dickson's pupils more especially associated with the University of Chicago, contributed to this formal modular theory of invariants. I made of it one of the main topics in my Jubilee Memoir published in Italy in 1950. The algebraic and modular theories were eventually coördinated, to the advantage of both.

The idea has persisted that the newer theories are not the only legitimate interests of the research worker. He will also wish to make contributions to the older, classical developments of his subject. Mindful of this principle, I developed a theory of linear invariant forms, (invariant elements), in a domain defined in terms of the parametric coefficients of the transformations. Combined with Hilbert's celebrated Lemma of the basis, this departure led to a new proof of the finiteness of the algebraic invariant systems, a theorem first stated by Cayley, (Gordan's Theorem). The method of elements was also applied to formal modular invariants and to differential invariants. The theorem of the finiteness of differential systems was not attempted. The present writer also contributed a series of articles on the listing of the systems of concomitants of ternary connexes, where the general method was the symbolical method of Clebsch and Gordan. Some fundamental extensions of theory were required in this program. A noted mathematician once expressed some misgivings as to whether a famous paper by Emmy Noether, in which the algebraic concomitants of the ternary quartic were definitively listed, was worth the great amount of exacting work which was required. I resolved this question affirmatively by showing that if we use a four-parameter subgroup instead of the general linear group itself, and particularize the $(u)$ set into any cogredient $(f(x))$ set, the complicated connexes in the fundamental system go over into covariants. Not even the resulting covariant system of the conic-form is yet known.

Still another new departure was the writer's theory of inverse processes in which what is assumed to be given is the invariant and what is required to be found is the transformation. This formulation throws much light on the functional invariants of infinitesimal transformations as used by Sophus Lie, and upon the theory of connections. It also effects certain classifications of functions. It furnished, when associated with results due to Carl Stormer of Norway, a computation and mapping of the Earth's lines of magnetic force.

Although invariant theory is well established as an independent field of thought in pure mathematics, it is gratifying that, in a half century, it has penetrated, or become relevant to, almost all mathematical disciplines, so that it is also evidently of importance in the realm of applications. In this realm the writer has contributed papers on invariants in the theory of orbits, in other physical theories, and in biomathematics. In the latter case it was shown how mathematics promotes our knowledge even of the complacent trilobite of the silurian beach. A theory of Saturn's rings, simpler than that advanced in the school of thought of James C. Maxwell was published by the present writer in 1935. Bode's law, for the respective distances of the planets from the Sun, was proved and generalized.

These latter results were possible after I had given, under the theory of orbits, a proof and generalization, the only proof ever achieved by inductive processes, of the law of inverse squares of Sir Isaac Newton, for gravitation. The memoir containing this result (1933), is among several which I published in the journal Annali della Scuola Normale Superiore, of Pisa, Italy. This journal was then printed in Bologna at the editorial and printing establishment of Signor Nicola Zanichelli. That the Newtonian law is a particular case, and that it was generalized effectively, changed traditional concepts of several types of orbital motion, although it did not alter the concept of the law of inverse squares as the principle that determines the motions of the large planets.

Much of natural philosophy is a philosophy of change or can be so interpreted readily. The differential calculus was the answer of mathematics to this principle in nature. Where there is change that is at all systematic, however, there will be something that remains unaltered during change. This is the invariant. It may be a number, like the length of the year; a function, as the discriminant of a binary quantic; a geometric figure, as the area in Kepler's law of areas; a relation, as any invariant equation; or a property, like the energy conserved in a specific mechanical system. One can say that the theory of invariants is naturally nearer to many characteristic concepts of natural phenomena than some of the other branches of mathematics.

The most complete bibliography of the writings herein mentioned is to be found in the encyclopedia of scientific publications which was a project of the Poggendorf Bureau in Germany. Through recent years the printing of this encyclopedia has been discontinued. It should be revived.

Lansdowne, Pa.

Information please:

In the November-December 1954 issue of MATHEMATICS MAGAZINE page 84 *So You Think You Can Count* is mention of *The Soroban Made Easy* by Miss Denslow.

Would you please tell me where this publication may be obtained, also, what company is manufacturing a soroban as described?

Ernest Weber

Send information to: Western Mich. College of Education, Kalamazoo 45, Mich., and to R. E. Ducotey, 11 Earney Mill Road, Newark, Delaware.

## OUR CONTRIBUTORS

(Continued from back of contents.)

*T. K. Pan* was born in Wenchow, China in 1904, and received his formal education at the University of Nanking (B.S.'27) and at the University of California (M.A.'33, Ph.D.'49). After teaching at Nanking for some twenty years, he returned to California and taught at the Berkeley Campus for six years before being appointed on the University of Oklahoma faculty, where he was made an Associate Professor in 1954. His specialty is Riemannian differential geometry. (*A REMARK ON THE QUOTIENT LAW OF TENSORS,* March-April.)

*C. D. Smith* has served on the editorial staff of this magazine for many years. Educated at Mississippi College (A.B.'15) and at the University of Chicago and Iowa (M.S.'25, Ph.D.'27) he taught in the high schools of Mississippi and Louisiana and then became department head at Louisiana College from 1919-29. After teaching at the Mississippi A. & M. College and at Mississippi State College, he joined the faculty of the University of Alabama where he was promoted to a Professorship in Statistics in 1953. (*TCHEBYCHEFF INEQUALITIES AS A BASIS FOR STATISTICAL TESTS,* March-April)

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*Comment by Becker . . . .*

Dear Prof. James:

I congratulate you on the Nov.-Dec. issue, as one of the most practically useful you ever edited. We used to teach Euclid's algorithm at Mare I. Trainee School; and H. S. Clair's article should help me in factoring large numbers, often a necessary evil in Diophantine Analysis. Corp. Adler's on the Abacus recalls how the grinning young Japanese beat the machine, while many G.I.'s cheered him on, then collected their bets! This ties in with L. E. Diamond's letter, if we want mathematics to win some of the popularity of athletics, we should have mathematics contests made interesting to spectators, with the same publicity buildup. The many rival systems of electric circuit analysis could contend in speed, etc. As to the X-Pone, it should be a God-send to electrical students, who even when they get their digits right, are prone to miss the decimal point 5 or 10 places! M. Kline's article was a real scoop!

H. W. Becker